Beginning and Intermediate Algebra

Chapter 9: Quadratics

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Chapter 9: Quadratics

9.1

Quadratics - Solving with Radicals

Here we look at equations that have roots in the problem. As you might expect, to clear a root we can raise both sides to an exponent. So to clear a square root we can rise both sides to the second power. To clear a cubed root we can raise both sides to a third power. There is one catch to solving a problem with roots in it, sometimes we end up with solutions that don't actually work in the equation. This will only happen if the index on the root is even, and it won't happen all the time. So for these problems it will be required that we check our answer in the original problem. If a value does not work it is called an extraneous solution and not included in the final solution.

When solving a radical problem with an even index: check answers!

Example 1.

$$\begin{array}{rl}
\sqrt{7x+2}=4 & \text{Even index! We will have to check answers}\\
(\sqrt{7x+2})^2=4^2 & \text{Square both sides, simplify exponents}\\
7x+2=16 & \text{Solve}\\
\underline{-2\quad -2} & \text{Subtract 2 from both sides}\\
7x=14 & \text{Divide both sides by 7}\\
\overline{7}\quad \overline{7} & \\
x=2 & \text{Need to check answer in original problem}\\
\sqrt{7(2)+2}=4 & \text{Multiply}\\
\sqrt{14+2}=4 & \text{Add}\\
\sqrt{16}=4 & \text{Square root}\\
4=4 & \text{True! It works!}\\
x=2 & \text{Our Solution}
\end{array}$$

Example 2.

$$\begin{array}{rl}
\sqrt[3]{x-1}=-4 & \text{Odd index, we don}'t\text{ need to check answer}\\
(\sqrt[3]{x-1})^3=(-4)^3 & \text{Cube both sides, simplify exponents}
\end{array}$$

$$\begin{aligned} x-1 &= -64 && \text{Solve}\\ \underline{+1 \quad} & \underline{\quad +1} && \text{Add 1 to both sides}\\ x &= -63 && \text{Our Solution}\end{aligned}$$

Example 3.

$$\begin{aligned} \sqrt[4]{3x+6} &= -3 && \text{Even index! We will have to check answers}\\ (\sqrt[4]{3x+6}) &= (-3)^4 && \text{Rise both sides to fourth power}\\ 3x+6 &= 81 && \text{Solve}\\ \underline{-6} & \underline{\ -6} && \text{Subtract 6 from both sides}\\ 3x &= 75 && \text{Divide both sides by 3}\\ \overline{3} & \ \ \overline{3} && \\ x &= 25 && \text{Need to check answer in original problem}\\ \sqrt[4]{3(25)+6} &= -3 && \text{Multiply}\\ \sqrt[4]{75+6} &= -3 && \text{Add}\\ \sqrt[4]{81} &= -3 && \text{Take root}\\ 3 &= -3 && \text{False, extraneous solution}\\ \text{No Solution} & && \text{Our Solution}\end{aligned}$$

If the radical is not alone on one side of the equation we will have to solve for the radical before we raise it to an exponent

Example 4.

$$\begin{aligned} x+\sqrt{4x+1} &= 5 && \text{Even index! We will have to check solutions}\\ \underline{-x \qquad\qquad} & \underline{\ \ -x} && \text{Isolate radical by subtracting } x \text{ from both sides}\\ \sqrt{4x+1} &= 5-x && \text{Square both sides}\\ (\sqrt{4x+1})^2 &= (5-x)^2 && \text{Evaluate exponents, recal } (a-b)^2 = a^2-2ab+b^2\\ 4x+1 &= 25-10x+x^2 && \text{Re}-\text{order terms}\\ 4x+1 &= x^2-10x+25 && \text{Make equation equal zero}\\ \underline{-4x-1 \quad} & \underline{\quad -4x \quad -1} && \text{Subtract } 4x \text{ and 1 from both sides}\\ 0 &= x^2-14x+24 && \text{Factor}\\ 0 &= (x-12)(x-2) && \text{Set each factor equal to zero}\\ x-12=0 \ \text{ or } \ x-2 &= 0 && \text{Solve each equation}\\ \underline{+12+12} \qquad \underline{+2} & \underline{+2} && \\ x=12 \ \text{ or } \ x &= 2 && \text{Need to check answers in original problem}\end{aligned}$$

$$
\begin{aligned}
(12)+\sqrt{4(12)+1}&=5 && \text{Check } x=5 \text{ first}\\
12+\sqrt{48+1}&=5 && \text{Add}\\
12+\sqrt{49}&=5 && \text{Take root}\\
12+7&=5 && \text{Add}\\
19&=5 && \text{False, extraneous root}
\end{aligned}
$$

$$
\begin{aligned}
(2)+\sqrt{4(2)+1}&=5 && \text{Check } x=2\\
2+\sqrt{8+1}&=5 && \text{Add}\\
2+\sqrt{9}&=5 && \text{Take root}\\
2+3&=5 && \text{Add}\\
5&=5 && \text{True! It works}\\
\\
x&=2 && \text{Our Solution}
\end{aligned}
$$

The above example illustrates that as we solve we could end up with an x^2 term or a quadratic. In this case we remember to set the equation to zero and solve by factoring. We will have to check both solutions if the index in the problem was even. Sometimes both values work, sometimes only one, and sometimes neither works.

If there is more than one square root in a problem we will clear the roots one at a time. This means we must first isolate one of them before we square both sides.

Example 5.

$$
\begin{aligned}
\sqrt{3x-8}-\sqrt{x}&=0 && \text{Even index! We will have to check answers}\\
+\sqrt{x}\ &+\sqrt{x} && \text{Isolate first root by adding } \sqrt{x} \text{ to both sides}\\
\sqrt{3x-8}&=\sqrt{x} && \text{Square both sides}\\
(\sqrt{3x-8})^2&=(\sqrt{x})^2 && \text{Evaluate exponents}\\
3x-8&=x && \text{Solve}\\
\underline{-3x\quad}&\underline{\quad-3x} && \text{Subtract } 3x \text{ from both sides}\\
-8&=-2x && \text{Divide both sides by } -2\\
\overline{-2}\quad&\quad\overline{-2}\\
4&=x && \text{Need to check answer in original}\\
\sqrt{3(4)-8}-\sqrt{4}&=0 && \text{Multiply}\\
\sqrt{12-8}-\sqrt{4}&=0 && \text{Subtract}\\
\sqrt{4}-\sqrt{4}&=0 && \text{Take roots}\\
2-2&=0 && \text{Subtract}\\
0&=0 && \text{True! It works}
\end{aligned}
$$

$$x=4 \quad \text{Our Solution}$$

When there is more than one square root in the problem, after isolating one root and squaring both sides we may still have a root remaining in the problem. In this case we will again isolate the term with the second root and square both sides. When isolating, we will isolate the *term* with the square root. This means the square root can be multiplied by a number after isolating.

Example 6.

$$\begin{array}{rl}
\sqrt{2x+1}-\sqrt{x}=1 & \text{Even index! We wil have to check answers} \\
\underline{+\sqrt{x}+\sqrt{x}} & \text{Isolate first root by adding } \sqrt{x} \text{ to both sides} \\
\sqrt{2x+1}=\sqrt{x}+1 & \text{Square both sides} \\
(\sqrt{2x+1})^2=(\sqrt{x}+1)^2 & \text{Evaluate exponents, recall } (a+b)^2=a^2+2ab+b^2 \\
2x+1=x+2\sqrt{x}+1 & \text{Isolate the term with the root} \\
\underline{-x-1-x\qquad -1} & \text{Subtract } x \text{ and 1 from both sides} \\
x=2\sqrt{x} & \text{Square both sides} \\
(x)^2=(2\sqrt{x})^2 & \text{Evaluate exponents} \\
x^2=4x & \text{Make equation equal zero} \\
-4x-4x & \text{Subtract } x \text{ from both sides} \\
x^2-4x=0 & \text{Factor} \\
x(x-4)=0 & \text{Set each factor equal to zero} \\
x=0 \ \text{ or } \ x-4=0 & \text{Solve} \\
\underline{+4+4} & \text{Add 4 to both sides of second equation} \\
x=0 \ \text{ or } \ x=4 & \text{Need to check answers in original}
\end{array}$$

$$\begin{array}{rl}
\sqrt{2(0)+1}-\sqrt{(0)}=1 & \text{Check } x=0 \text{ first} \\
\sqrt{1}-\sqrt{0}=1 & \text{Take roots} \\
1-0=1 & \text{Subtract} \\
1=1 & \text{True! It works}
\end{array}$$

$$\begin{array}{rl}
\sqrt{2(4)+1}-\sqrt{(4)}=1 & \text{Check } x=4 \\
\sqrt{8+1}-\sqrt{4}=1 & \text{Add} \\
\sqrt{9}-\sqrt{4}=1 & \text{Take roots} \\
3-2=1 & \text{Subtract} \\
1=1 & \text{True! It works}
\end{array}$$

$$x=0 \ \text{ or } \ 4 \quad \text{Our Solution}$$

Example 7.

$$
\begin{array}{rl}
\sqrt{3x+9}-\sqrt{x+4}=-1 & \text{Even index! We will have to check answers}\\
\underline{+\sqrt{x+4}+\sqrt{x+4}} & \text{Isolate the first root by adding } \sqrt{x+4}\\
\sqrt{3x+9}=\sqrt{x+4}-1 & \text{Square both sides}\\
(\sqrt{3x+9})^2=(\sqrt{x+4}-1)^2 & \text{Evaluate exponents}\\
3x+9=x+4-2\sqrt{x+4}+1 & \text{Combine like terms}\\
3x+9=x+5-2\sqrt{x+4} & \text{Isolate the term with radical}\\
\underline{-x-5-x-5}\qquad\qquad\quad & \text{Subtract } x \text{ and } 5 \text{ from both sides}\\
2x+4=-2\sqrt{x+4} & \text{Square both sides}\\
(2x+4)^2=(-2\sqrt{x+4})^2 & \text{Evaluate exponents}\\
4x^2+16x+16=4(x+4) & \text{Distirbute}\\
4x^2+16x+16=4x+16 & \text{Make equation equal zero}\\
\underline{-4x-16-4x-16} & \text{Subtract } 4x \text{ and } 16 \text{ from both sides}\\
4x^2+12x=0 & \text{Factor}\\
4x(x+3)=0 & \text{Set each factor equal to zero}\\
4x=0 \ \text{ or } \ x+3=0 & \text{Solve}\\
\overline{4}\quad\overline{4}\qquad\quad \underline{-3-3} & \\
x=0 \ \text{ or } \ x=-3 & \text{Check solutions in original}\\
 & \\
\sqrt{3(0)+9}-\sqrt{(0)+4}=-1 & \text{Check } x=0 \text{ first}\\
\sqrt{9}-\sqrt{4}=-1 & \text{Take roots}\\
3-2=-1 & \text{Subtract}\\
1=-1 & \text{False, extraneous solution}\\
 & \\
\sqrt{3(-3)+9}-\sqrt{(-3)+4}=-1 & \text{Check } x=-3\\
\sqrt{-9+9}-\sqrt{(-3)+4}=-1 & \text{Add}\\
\sqrt{0}-\sqrt{1}=-1 & \text{Take roots}\\
0-1=-1 & \text{Subtract}\\
-1=-1 & \text{True! It works}\\
 & \\
x=-3 & \text{Our Solution}
\end{array}
$$

Practice - Solving with Radicals

Solve.

1) $\sqrt{2x+3}-3=0$

2) $\sqrt{5x+1}-4=0$

3) $\sqrt{6x-5}-x=0$

4) $\sqrt{x+2}-\sqrt{x}=2$

5) $3+x=\sqrt{6x+13}$

6) $x-2=\sqrt{7-x}$

7) $\sqrt{3-3x}-1=2x$

8) $\sqrt{2x+2}=3+\sqrt{2x-1}$

9) $\sqrt{4x+5}-\sqrt{x+4}=2$

10) $\sqrt{3x+4}-\sqrt{x+2}=2$

11) $\sqrt{2x+4}-\sqrt{x+3}=1$

12) $\sqrt{7x+2}-\sqrt{3x+6}=1$

13) $\sqrt{2x+6}-\sqrt{x+4}=2$

14) $\sqrt{4x-3}-\sqrt{3x+1}=1$

15) $\sqrt{6-2x}-\sqrt{2x+3}=3$

16) $\sqrt{2-3x}-\sqrt{3x+7}=3$

9.2

Quadratics - Solving with Exponents

Another type of equation we can solve is one with exponents. As you might expect we can clear exponents by using roots. This is done with very few unexpected results with the exponent is odd. We solve these problems very straight forword using the odd root property

Odd Root Property: if $a^n = b$, then $a = \sqrt[n]{b}$ when n is odd

Example 8.

$$\begin{aligned} x^5 &= 32 && \text{Use odd root property} \\ \sqrt[5]{x^5} &= \sqrt[5]{32} && \text{Simplify roots} \\ x &= 2 && \text{Our Solution} \end{aligned}$$

However, when the exponent is even we will have two results from taking an even root of both sides. One will be positive and one will be negative. This is because both $3^2 = 9$ and $(-3)^2 = 9$. so when solving $x^2 = 9$ we will have two solutions, one positive and one negative: $x = 3 \text{ and } -3$

Even Root Property: if $a^n = b$, then $a = \pm\sqrt[n]{b}$ when n is even

Example 9.

$$
\begin{aligned}
x^4 &= 16 && \text{Use even root property } (\pm) \\
\sqrt[4]{x^4} &= \pm\sqrt[4]{16} && \text{Simplify roots} \\
x &= \pm 2 && \text{Our Solution}
\end{aligned}
$$

Example 10.

$$
\begin{aligned}
(2x+4)^2 &= 36 && \text{Use even root property } (\pm) \\
\sqrt{(2x+4)^2} &= \pm\sqrt{36} && \text{Simplify roots} \\
2x+4 &= \pm 6 && \text{To avoid sign errors we need two equations} \\
2x+4=6 \quad \text{or} \quad 2x+4 &= -6 && \text{One equation for } +\text{, one equation for } - \\
\underline{-4\,-4} \qquad\quad \underline{-4 \quad -4} & && \text{Subtract 4 from both sides} \\
2x=6 \quad \text{or} \quad 2x &= -10 && \text{Divide both sides by 2} \\
\overline{2} \;\; \overline{2} \qquad\;\; \overline{2} \quad\;\; & \overline{2} && \\
x=3 \quad \text{or} \quad x &= -5 && \text{Our Solutions}
\end{aligned}
$$

In the previous example we needed to equations to simplify because when we took the root, our solution were two rational numbers, 6 and -6. If the roots did not simplify to rational numbers we can keep the $\pm$ in the equation.

Example 11.

$$
\begin{aligned}
(6x-9)^2 &= 45 && \text{Use even root property } (\pm) \\
\sqrt{(6x-9)^2} &= \pm\sqrt{45} && \text{Simplify roots} \\
6x-9 &= \pm 3\sqrt{5} && \text{Use one equation because root did not simplify to rational} \\
\underline{+9 \;\; +9} & && \text{Add 9 to both sides} \\
6x &= 9\pm 3\sqrt{5} && \text{Divide both sides by 6} \\
\overline{6} \quad & \overline{\quad 6 \quad} && \\
x &= \frac{9\pm 3\sqrt{5}}{6} && \text{Simplify, divide each term by 3} \\
x &= \frac{3\pm\sqrt{5}}{2} && \text{Our Solution}
\end{aligned}
$$

When solving with exponents, it is important to first isolate the part with the exponent before taking any roots.

Example 12.

$$
\begin{array}{rl}
(x+4)^3 - 6 = 119 & \text{Isolate part with exponent} \\
\underline{+6 \quad +6} & \\
(x+4)^3 = 125 & \text{Use odd root property} \\
\sqrt[3]{(x+4)^3} = \sqrt{125} & \text{Simplify roots} \\
x + 4 = 5 & \text{Solve} \\
\underline{-4 - 4} & \text{Subtract 4 from both sides} \\
x = 1 & \text{Our Solution}
\end{array}
$$

Example 13.

$$
\begin{array}{rl}
(6x+1)^2 + 6 = 10 & \text{Isolate part with exponent} \\
\underline{-6 - 6} & \text{Subtract 6 from both sides} \\
(6x+1)^2 = 4 & \text{Use even root property } (\pm) \\
\sqrt{(6x+1)^2} = \pm\sqrt{4} & \text{Simplify roots} \\
6x + 1 = \pm 2 & \text{To avoid sign errors, we need two equations} \\
6x + 1 = 2 \text{ or } 6x + 1 = -2 & \text{Solve each equation} \\
\underline{-1 - 1} \qquad \underline{-1 \quad -1} & \text{Subtract 1 from both sides} \\
6x = 1 \text{ or } 6x = -3 & \text{Divide both sides by 6} \\
\overline{6} \ \overline{6} \qquad \overline{6} \quad \overline{6} & \\
x = \dfrac{1}{6} \text{ or } x = -\dfrac{1}{2} & \text{Our Solution}
\end{array}
$$

When our exponents are a fraction we will need to first convert the fractional exponent into a radical expression to solve. Recall that $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. Once we have done this we can clear the exponent using either the even ($\pm$) or odd root property. Then we can clear the radical by raising both sides to an exponent (remember to check answers if the index is even).

Example 14.

$$
\begin{array}{rl}
(4x+1)^{\frac{2}{5}} = 9 & \text{Rewrite as } a \text{ radical expression} \\
(\sqrt[5]{4x+1})^2 = 9 & \text{Clear exponent first with even root property } (\pm) \\
\sqrt{(\sqrt[5]{4x+1})^2} = \pm\sqrt{9} & \text{Simplify roots}
\end{array}
$$

$$
\begin{array}{rl}
\sqrt[5]{4x+1}=\pm 3 & \text{Clear radical by raising both sides to 5th power}\\
(\sqrt[5]{4x+1})^5=(\pm 3)^5 & \text{Simplify exponents}\\
4x+1=243 & \text{Solve}\\
\underline{-1\quad -1} & \text{Subtract 1 from both sides}\\
4x=242 & \text{Divide both sides by 4}\\
\overline{4}\quad \overline{4} & \\
x=\dfrac{121}{2} & \text{Our Solution}
\end{array}
$$

Example 15.

$$
\begin{array}{rl}
(3x-2)^{\frac{3}{4}}=64 & \text{Rewrite as radical expression}\\
(\sqrt[4]{3x-2})^3=64 & \text{Clear exponent first with odd root property}\\
\sqrt[3]{(\sqrt[4]{3x-2})^3}=\sqrt[3]{64} & \text{Simplify roots}\\
\sqrt[4]{3x-2}=4 & \text{Even Index! Check answers.}\\
(\sqrt[4]{3x-2})^4=4^4 & \text{Raise both sides to 4th power}\\
3x-2=256 & \text{Solve}\\
\underline{+2\quad +2} & \text{Add 2 to both sides}\\
3x=258 & \text{Divide both sides by 3}\\
\overline{3}\quad \overline{3} & \\
x=86 & \text{Need to check answer in radical form of problem}\\
(\sqrt[4]{3(86)-2})^3=64 & \text{Multiply}\\
(\sqrt[4]{258-2})^3=64 & \text{Subtract}\\
(\sqrt[4]{256})^3=64 & \text{Evaluate root}\\
4^3=64 & \text{Evaluate exponent}\\
64=64 & \text{True! It works}\\
x=86 & \text{Our Solution}
\end{array}
$$

With rational exponents it is very helpful to convert to radical form to be able to see if we need a $\pm$ because we used the even root property, or to see if we need to check our answer because there was an even root in the problem. When checking we will usually want to check in the radical form as it will be easier to evaluate.

Practice - Solving with Exponents

Solve.

1) $x^2 = 75$

2) $x^3 = -8$

3) $x^2 + 5 = 13$

4) $4x^3 - 2 = 106$

5) $3x^2 + 1 = 73$

6) $(x - 4)^2 = 49$

7) $(x + 2)^5 = -243$

8) $(5x + 1)^4 = 16$

9) $(2x + 5)^3 - 6 = 21$

10) $(2x + 1)^2 + 3 = 21$

11) $(x - 1)^{\frac{2}{3}} = 16$

12) $(x - 1)^{\frac{3}{2}} = 8$

13) $(2 - x)^{\frac{3}{2}} = 27$

14) $(2x + 3)^{\frac{4}{3}} = 16$

15) $(2x - 3)^{\frac{2}{3}} = 4$

16) $(x + 3)^{-\frac{1}{3}} = 4$

17) $(x + \frac{1}{2})^{-\frac{2}{3}} = 4$

18) $(x - 1)^{-\frac{5}{3}} = 32$

19) $(x - 1)^{-\frac{5}{2}} = 32$

20) $(x + 3)^{\frac{3}{2}} = -8$

21) $(3x - 2)^{\frac{4}{5}} = 16$

22) $(2x + 3)^{\frac{3}{2}} = 27$

23) $(4x + 2)^{\frac{3}{5}} = -8$

24) $(3 - 2x)^{\frac{4}{3}} = -81$

9.3

Quadratics - Complete the Square

When solving quadratic equations in the past we have used factoring to solve for our variable. This is exactly what is done in the next example.

Example 16.

$$\begin{array}{rcrl} & & x^2+5x+6=0 & \text{Factor} \\ & & (x+3)(x+2)=0 & \text{Set each factor equal to zero} \\ x+3=0 & \text{or} & x+2=0 & \text{Solve each equation} \\ \underline{-3-3} & & \underline{-2-2} & \\ x=-3 & \text{or} & x=-2 & \text{Our Solutions} \end{array}$$

However, the problem with factoring is all equations cannot be factored. Consider the following equation: $x^2-2x-7=0$. The equation cannot be factored, however there are two solutions to this equation, $1+2\sqrt{2}$ and $1-2\sqrt{2}$. To find these two solutions we will use a method known as completing the square. When completing the square we will change the quadratic into a perfect square which can easily be solved with the square root property. The next example reviews the square root property.

Example 17.

$$\begin{array}{rl} (x+5)^2=18 & \text{Square root of both sides} \\ \sqrt{(x+5)^2}=\pm\sqrt{18} & \text{Simplify each radical} \\ x+5=\pm 3\sqrt{2} & \text{Subtract 5 from both sides} \\ \underline{-5\quad -5\qquad} & \end{array}$$

$$x = -5 \pm 3\sqrt{2} \qquad \text{Our Solution}$$

To complete the square, or make our problem into the form of the previous example, we will be searching for the third term in a trinomial. If a quadratic is of the form $x^2 + bx + c$, and a perfect square, the third term, c, can be easily found by the formula $\left(\frac{1}{2} \cdot b\right)^2$. This is shown in the following examples, where we find the number that completes the square and then factor the perfect square.

Example 18.

$$\begin{aligned}
x^2 + 8x + c \quad & c = \left(\frac{1}{2} \cdot b\right)^2 \text{ and our } b = 8\\
\left(\frac{1}{2} \cdot 8\right)^2 = 4^2 = 16 \quad & \text{The third term to complete the square is } 16\\
x^2 + 8x + 16 \quad & \text{Our equation as } a \text{ perfect square, factor}\\
(x+4)^2 \quad & \text{Our Solution}
\end{aligned}$$

Example 19.

$$\begin{aligned}
x^2 - 11x + c \quad & c = \left(\frac{1}{2} \cdot b\right)^2 \text{ and our } b = 8\\
\left(\frac{1}{2} \cdot 7\right)^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4} \quad & \text{The third term to complete the square is } \frac{49}{4}\\
x^2 - 11x + \frac{49}{4} \quad & \text{Our equation as } a \text{ perfect square, factor}\\
\left(x - \frac{7}{2}\right)^2 \quad & \text{Our Solution}
\end{aligned}$$

Example 20.

$$x^2 + \frac{5}{3}x + c \qquad c = \left(\frac{1}{2} \cdot b\right)^2 \text{ and our } b = 8$$

$$\left(\frac{1}{2}\cdot\frac{5}{3}\right)^2=\left(\frac{5}{6}\right)^2=\frac{25}{36}\quad \text{The third term to complete the square is } \frac{25}{36}$$

$$x^2+\frac{5}{3}x+\frac{25}{36}\quad \text{Our equation as } a \text{ perfect square, factor}$$

$$\left(x+\frac{5}{6}\right)^2\quad \text{Our Solution}$$

The process in the previous examples, combined with the even root property, is used to solve quadratic equations by completing the square. The following five steps describe the process used to complete the square, along with an example to demonstrate each step.

Problem	$3x^2+18x-6=0$
1. Separate constant term from variables	$\underline{\quad +6+6}$ $3x^2+18x \quad =6$
2. Divide each term by a	$\frac{3}{3}x^2+\frac{18}{3}x \quad =\frac{6}{3}$ $x^2+6x \quad =2$
3. Find value to complete the square: $\left(\frac{1}{2}\cdot b\right)^2$	$\left(\frac{1}{2}\cdot 6\right)^2=3^2=9$
4. Add to both sides of equation	$x^2+6x \quad =2$ $\underline{\quad +9+9}$ $x^2+6x+9=11$
5. Factor	$(x+3)^2=11$
Solve by even root property	$\sqrt{(x+3)^2}=\pm\sqrt{11}$ $x+3=\pm\sqrt{11}$ $\underline{-3\quad -3}$ $x=-3\pm\sqrt{11}$

The advantage of this method is it can be used to solve any quadratic equation. The following examples show how completing the square can give us rational solution, irrational solutions, and even complex solutions.

Example 21.

$$2x^2+20x+48=0\quad \text{Separate constant term from varaibles}$$

$$\begin{array}{rl}
\underline{-48-48} & \text{Subtract 24} \\
2x^2+20x \quad = -48 & \text{Divide by } a \text{ or } 2 \\
\overline{2} \quad \overline{2} \quad \quad \overline{2} & \\
x^2+10x \quad = -24 & \text{Find number to complete the square: } \left(\frac{1}{2}\cdot b\right)^2 \\
\left(\frac{1}{2}\cdot 10\right)^2 = 5^2 = 25 & \text{Add 25 to both sides of the equation} \\
x^2+10x \quad = -24 & \\
\underline{+25 \quad +25} & \\
x^2+10x+25 = 1 & \text{Factor} \\
(x+5)^2 = 1 & \text{Solve with even root property} \\
\sqrt{(x+5)^2} = \pm\sqrt{1} & \text{Simplify roots} \\
x+5 = \pm 1 & \text{Subtract 5 from both sides} \\
\underline{-5-5} & \\
x = -5 \pm 1 & \text{Evaluate} \\
x = -4 \text{ or } -6 & \text{Our Solution}
\end{array}$$

Example 22.

$$\begin{array}{rl}
x^2-3x-2=0 & \text{Separate constant from variables} \\
\underline{+2+2} & \text{Add 2 to both sides} \\
x^2-3x \quad = 2 & \text{No } a\text{, find number to complete the square } \left(\frac{1}{2}\cdot b\right)^2 \\
\left(\frac{1}{2}\cdot 3\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} & \text{Add } \frac{9}{4} \text{ to both sides,} \\
\frac{2}{1}\left(\frac{4}{4}\right) + \frac{9}{4} = \frac{8}{4} + \frac{9}{4} = \frac{17}{4} & \text{Need common denominator (4) on right} \\
x^2-3x+\frac{9}{4} = \frac{8}{4} + \frac{9}{4} = \frac{17}{4} & \text{Factor} \\
\left(x-\frac{3}{2}\right)^2 = \frac{17}{4} & \text{Solve using the even root property} \\
\sqrt{\left(x-\frac{3}{2}\right)^2} = \pm\sqrt{\frac{17}{4}} & \text{Simplify roots} \\
x-\frac{3}{2} = \frac{\pm\sqrt{17}}{2} & \text{Add } \frac{3}{2} \text{ to both sides,}
\end{array}$$

$$\begin{array}{ll} \underline{+\frac{3}{2}+\frac{3}{2}} & \text{we already have } a \text{ common denominator} \\ x=\frac{3\pm\sqrt{17}}{2} & \text{Our Solution} \end{array}$$

Example 23.

$$\begin{array}{rl} 3x^2=2x-7 & \text{Separate the constant from the variables} \\ \underline{-2x-2x} & \text{Subtract } 2x \text{ from both sides} \\ \frac{3x^2}{3}-\frac{2x}{3}=\frac{-7}{3} & \text{Divide each term by } a \text{ or } 3 \\ x^2-\frac{2}{3}x \quad =-\frac{7}{3} & \text{Find the number to complete the square } \left(\frac{1}{2}\cdot b\right)^2 \\ \left(\frac{1}{2}\cdot\frac{2}{3}\right)^2=\left(\frac{1}{3}\right)^2=\frac{1}{9} & \text{Add to both sides,} \\ -\frac{7}{3}\left(\frac{3}{3}\right)+\frac{1}{9}=\frac{-21}{3}+\frac{1}{9}=\frac{-20}{9} & \text{get common denominator on right} \\ x^2-\frac{2}{3}x+\frac{1}{3}=-\frac{20}{9} & \text{Factor} \\ \left(x-\frac{1}{3}\right)^2=-\frac{20}{9} & \text{Solve using the even root property} \\ \sqrt{\left(x-\frac{1}{3}\right)^2}=\pm\sqrt{\frac{-20}{9}} & \text{Simplify roots} \\ x-\frac{1}{3}=\frac{\pm 2i\sqrt{5}}{3} & \text{Add } \frac{1}{3} \text{ to both sides,} \\ \underline{+\frac{1}{3}+\frac{1}{3}} & \text{Already have common denominator} \\ x=\frac{1\pm 2i\sqrt{5}}{3} & \text{Our Solution} \end{array}$$

As several of the examples have shown, when solving by completing the square we will often need to use fraction and be comfortable finding common denominators and adding fractions together. Once we get comfortable solving by completing the square and using the five steps, any quadratic equation can be easily solved.

Practice - Complete the Square

Find the value that completes the square and then rewrite as a perfect square.

1) $x^2 - 30x + \_\_$

2) $a^2 - 24a + \_\_$

3) $m^2 - 36m + \_\_$

4) $x^2 - 34x + \_\_$

5) $x^2 - 15x + \ \_\_$

6) $r^2 - \frac{1}{9}r + \_\_$

7) $y^2 - y + \_\_$

8) $p^2 - 17p + \_\_$

Solve each equation by completing the square.

9) $x^2 - 16x + 55 = 0$

10) $n^2 - 8n - 12 = 0$

11) $v^2 - 8v + 45 = 0$

12) $b^2 + 2b + 43 = 0$

13) $6x^2 + 12x + 63 = 0$

14) $3x^2 - 6x + 47 = 0$

15) $5k^2 - 10k + 48 = 0$

16) $8a^2 + 16a - 1 = 0$

17) $x^2 + 10x - 57 = 4$

18) $p^2 - 16p - 52 = 0$

19) $n^2 - 16n + 67 = 4$

20) $m^2 - 8m - 3 = 6$

21) $2x^2 + 4x + 38 = -6$

22) $6r^2 + 12r - 24 = -6$

23) $8b^2 + 16b - 37 = 5$

24) $6n^2 - 12n - 14 = 4$

25) $x^2 = -10x - 29$

26) $v^2 = 14v + 36$

27) $n^2 = -21 + 10n$

28) $a^2 - 56 = -10a$

29) $3k^2 + 9 = 6k$

30) $5x^2 = -26 + 10x$

31) $2x^2 + 63 = 8x$

32) $5n^2 = -10n + 15$

33) $p^2 - 8p = -55$

34) $x^2 + 8x + 15 = 8$

35) $7n^2 - n + 7 = 7n + 6n^2$

36) $n^2 + 4n = 12$

37) $13b^2 + 15b + 44 = -5 + 7b^2 + 3b$

38) $-3r^2 + 12r + 49 = -6r^2$

39) $5x^2 + 5x = -31 - 5x$

40) $8n^2 + 16n = 64$

41) $v^2 + 5v + 28 = 0$

42) $b^2 + 7b - 33 = 0$

43) $7x^2 - 6x + 40 = 0$

44) $4x^2 + 4x + 25 = 0$

45) $k^2 - 7k + 50 = 3$

46) $a^2 - 5a + 25 = 3$

47) $5x^2 + 8x - 40 = 8$

48) $2p^2 - p + 56 = -8$

49) $m^2 = -15 + 9m$

50) $n^2 - n = -41$

51) $8r^2 + 10r = -55$

52) $3x^2 - 11x = -18$

53) $5n^2 - 8n + 60 = -3n + 6 + 4n^2$

54) $4b^2 - 15b + 56 = 3b^2$

55) $-2x^2 + 3x - 5 = -4x^2$

56) $10v^2 - 15v = 27 + 4v^2 - 6v$

9.4

Quadratics - Quadratic Formula

The general from of a quadratic is $ax^2+bx+c=0$. We will now solve this formula for x by completing the square

Example 24.

$$\begin{array}{rl}
ax^2+bc+c=0 & \text{Separate constant from varaibles} \\
\underline{-c-c} & \text{Subtract } c \text{ from both sides} \\
\dfrac{ax^2}{a}+\dfrac{bx}{a} \quad =\dfrac{-c}{a} & \text{Divide each term by } a \\
x^2+\dfrac{b}{a}x \quad =\dfrac{-c}{a} & \text{Find the number that completes the square} \\
\left(\dfrac{1}{2}\cdot\dfrac{b}{a}\right)^2=\left(\dfrac{b}{2a}\right)^2=\dfrac{b^2}{4a^2} & \text{Add to both sides,} \\
\dfrac{b^2}{4a^2}-\dfrac{c}{a}\left(\dfrac{4a}{4a}\right)=\dfrac{b^2}{4a^2}-\dfrac{4ac}{4a^2}=\dfrac{b^2-4ac}{4a^2} & \text{Get common denominator on right} \\
x^2+\dfrac{b}{a}x+\dfrac{b^2}{4a^2}=\dfrac{b^2}{4a^2}-\dfrac{4\text{ac}}{4a^2}=\dfrac{b^2-4ac}{4a^2} & \text{Factor} \\
\left(x+\dfrac{b}{2a}\right)^2=\dfrac{b^2-4ac}{4a^2} & \text{Solve using the even root property} \\
\sqrt{\left(x+\dfrac{b}{2a}\right)^2}=\pm\sqrt{\dfrac{b^2-4ac}{4a^2}} & \text{Simplify roots} \\
x+\dfrac{b}{2a}=\dfrac{\pm\sqrt{b^2-4ac}}{2a} & \text{Subtract } \dfrac{b}{2a} \text{ from both sides}
\end{array}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Our Solution}$$

This solution is a very important one to us. As we solved a general equation by completing the square, we can use this formula to solve any quadratic equation. Once we identify what a, b, and c are in the quadratic, we can substitute those values into $x = \frac{-b \pm \sqrt{b^2 - 4\mathrm{ac}}}{2a}$ and we will get our two solution. This formula is known as the quadratic fromula

$$\textbf{Quadratic Formula: if } \boldsymbol{ax^2 + bx + c = 0} \textbf{ then } \boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

We can use the quadratic formula to solve any quadratic, this is shown in the following examples.

Example 25.

$$\begin{aligned}
x^2 + 3x + 2 &= 0 && a = 1, b = 3, c = 2, \text{use quadratic formula} \\
x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2(1)} && \text{Evaluate exponent and multiplication} \\
x &= \frac{-3 \pm \sqrt{9 - 8}}{2} && \text{Evaluate subtraction under root} \\
x &= \frac{-3 \pm \sqrt{1}}{2} && \text{Evaluate root} \\
x &= \frac{-3 \pm 1}{2} && \text{Evaluate } \pm \text{ to get two answers} \\
x &= \frac{-2}{2} \text{ or } \frac{-4}{2} && \text{Simplify fractions} \\
x &= -1 \text{ or } -2 && \text{Our Solution}
\end{aligned}$$

As we are solving using the quadratic formula, it is important to remember the equation must fist be equal to zero.

Example 26.

$$\begin{aligned}
25x^2 &= 30x + 11 && \text{First set equal to zero} \\
\underline{-30x - 11} & \underline{\;\; -30x - 11} && \text{Subtract } 30x \text{ and } 11 \text{ from both sides} \\
25x^2 - 30x - 11 &= 0 && a = 25, b = -30, c = -11, \text{use quadratic formula}
\end{aligned}$$

$$
\begin{array}{rl}
x = \dfrac{30 \pm \sqrt{(-30)^2 - 4(25)(-11)}}{2(25)} & \text{Evaluate exponet and multiplication} \\
x = \dfrac{30 \pm \sqrt{900 + 1100}}{50} & \text{Evaluate addition inside root} \\
x = \dfrac{30 \pm \sqrt{2000}}{50} & \text{Simplify root} \\
x = \dfrac{30 \pm 20\sqrt{5}}{50} & \text{Reduce fraction by dividing each term by 10} \\
x = \dfrac{3 \pm 2\sqrt{5}}{5} & \text{Our Solution}
\end{array}
$$

Example 27.

$$
\begin{array}{rl}
3x^2 + 4x + 8 = 2x^2 + 6x - 5 & \text{First set equation equal to zero} \\
\underline{-2x^2 - 6x + 5 \quad -2x^2 - 6x + 5} & \text{Subtract } 2x^2 \text{ and } 6x \text{ and add } 5 \\
x^2 - 2x + 13 = 0 & a = 1, b = -2, c = 13, \text{ use quadratic formula} \\
x = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(13)}}{2(1)} & \text{Evaluate exponent and multiplication} \\
x = \dfrac{2 \pm \sqrt{4 - 52}}{2} & \text{Evaluate subtraction inside root} \\
x = \dfrac{2 \pm \sqrt{-48}}{2} & \text{Simplify root} \\
x = \dfrac{2 \pm 4i\sqrt{3}}{2} & \text{Reduce fraction by dividing each term by 2} \\
x = 1 \pm 2i\sqrt{3} & \text{Our Solution}
\end{array}
$$

When we use the quadratic formula we don't necessarily get two unique answers. We can end up with only one solution if the square root simplifies to zero.

Example 28.

$$
\begin{array}{rl}
4x^2 - 12x + 9 = 0 & a = 4, b = -12, c = 9, \text{ use quadratic formula} \\
x = \dfrac{12 \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} & \text{Evaluate exponents and multiplication} \\
x = \dfrac{12 \pm \sqrt{144 - 144}}{8} & \text{Evaluate subtraction inside root} \\
x = \dfrac{12 \pm \sqrt{0}}{8} & \text{Evaluate root} \\
x = \dfrac{12 \pm 0}{8} & \text{Evaluate } \pm
\end{array}
$$

$$x = \frac{12}{8} \quad \text{Reduce fraction}$$
$$x = \frac{3}{2} \quad \text{Our Solution}$$

If a term is missing from the quadratic, we can still solve with the quadratic formula, we simply use zero for that term. The order is important, so if the term with x is missing, we have $b = 0$, if the constant term is missing, we have $c = 0$.

Example 29.

$$3x^2 + 7 = 0 \quad a = 3, b = 0\,(\text{missing term}), c = 7$$
$$x = \frac{-0 \pm \sqrt{0^2 - 4(3)(7)}}{2(3)} \quad \text{Evaluate exponnets and multiplication, zeros not needed}$$
$$x = \frac{\pm \sqrt{-84}}{6} \quad \text{Simplify root}$$
$$x = \frac{\pm 2i\sqrt{21}}{6} \quad \text{Reduce, dividing by 2}$$
$$x = \frac{\pm i\sqrt{21}}{3} \quad \text{Our Solution}$$

We have covered three different methods to use to solve a quadratic: factoring, complete the square, and the quadratic formula. It is important to be familiar with all three as each has its advantage to solving quadratics. The following table walks through a suggested process to decide which method would be best to use for solving a problem.

1. If it can easily factor, solve by factoring	$x^2 - 5x + 6 = 0$ $(x - 2)(x - 3) = 0$ $x = 2$ or $x = 3$
2. If $a = 1$ and b is even, complete the square	$x^2 + 2x = 4$ $\left(\frac{1}{2} \cdot 2\right)^2 = 1^2 = 1$ $x^2 + 2x + 1 = 5$ $(x + 1)^2 = 5$ $x + 1 = \pm \sqrt{5}$ $x = -1 \pm \sqrt{5}$
3. Otherwise, solve by the quadratic formula	$x^2 - 3x + 4 = 0$ $x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2(1)}$ $x = \frac{3 \pm i\sqrt{7}}{2}$

The above table is mearly a suggestion for deciding how to solve a quadtraict. Remember completing the square and quadratic formula will always work to solve any quadratic. Factoring only woks if the equation can be factored.

Practice - Quadratic Formula

Solve each equation with the quadratic formula.

1) $4a^2+6=0$

2) $3k^2+2=0$

3) $2x^2-8x-2=0$

4) $6n^2-1=0$

5) $2m^2-3=0$

6) $5p^2+2p+6=0$

7) $3r^2-2r-1=0$

8) $2x^2-2x-15=0$

9) $4n^2-36=0$

10) $3b^2+6=0$

11) $v^2-4v-5=-8$

12) $2x^2+4x+12=8$

13) $2a^2+3a+14=6$

14) $6n^2-3n+3=-4$

15) $3k^2-3k-4=7$

16) $4x^2-14=-2$

17) $7x^2+3x-16=-2$

18) $4n^2+5n=7$

19) $2p^2+6p-16=4$

20) $m^2+4m-48=-3$

21) $3n^2+3n=-3$

22) $3b^2-3=8b$

23) $2x^2=-7x+49$

24) $3r^2+4=-6r$

25) $5x^2=7x+7$

26) $6a^2=-5a+13$

27) $8n^2=-3n-8$

28) $6v^2=4+6v$

29) $2x^2+5x=-3$

30) $x^2=8$

31) $4a^2-64=0$

32) $2k^2+6k-16=2k$

33) $4p^2+5p-36=3p^2$

34) $12x^2+x+7=5x^2+5x$

35) $-5n^2-3n-52=2-7n^2$

36) $7m^2-6m+6=-m$

37) $7r^2-12=-3r$

38) $3x^2-3=x^2$

39) $2n^2-9=4$

40) $6b^2=b^2+7-b$

9.5

Quadratics - Equation from Solutions

Up to this point we have found the solutions to quadratics by a method such as factoring or completing the square. Here we will take our solutions and work backwards to find what quadratic goes with the solutions.

We will start with rational solutions. If we have rational solutions we can use factoring in reverse, we will set each solution equal to x and then make the equation equal zero by adding or subtracting. Once we have done this our expressions will become the factors of the quadratic.

Example 30.

$$\begin{array}{rl}
\text{The solutions are } 4 \text{ and } -2 & \text{Set each solution equal to } x \\
x = 4 \;\text{ or }\; x = -2 & \text{Make each equation equal zero} \\
\underline{-4\;-4} \quad \underline{+2\quad+2} & \text{Subtract 4 from first, add 2 to second} \\
x - 4 = 0 \;\text{ or }\; x + 2 = 0 & \text{These expressions are the factors} \\
(x-4)(x+2) = 0 & \text{FOIL} \\
x^2 + 2x - 4x - 8 & \text{Combine like terms} \\
x^2 - 2x - 8 = 0 & \text{Our Solution}
\end{array}$$

If one or both of the solutions are fractions we will clear the fractions by multiplying by the denominators.

Example 31.

$$\begin{array}{rl}
\text{The solution are } \frac{2}{3} \text{ and } \frac{3}{4} & \text{Set each solution equal to } x \\
x = \frac{2}{3} \;\text{ or }\; x = \frac{3}{4} & \text{Clear fractions by multiplying by denominators} \\
3x = 2 \;\text{ or }\; 4x = 3 & \text{Make each equation equal zero} \\
\underline{-2\;-2} \quad \underline{-3\;-3} & \text{Subtract 2 from the first, subtract 3 from the second} \\
3x - 2 = 0 \;\text{ or }\; 4x - 3 = 0 & \text{These expressions are the factors} \\
(3x-2)(4x-3) = 0 & \text{FOIL} \\
12x^2 - 9x - 8x + 6 = 0 & \text{Combine like terms} \\
12x^2 - 17x + 6 = 0 & \text{Our Solution}
\end{array}$$

If the solutions have radicals (or complex numbers) then we cannot use reverse factoring. In these cases we will use reverse completing the square. When there are radicals the solutions will always come in pairs, one with a plus, one with a minus, that can be combined into “one” solution using $\pm$. We will this set this solution equal to x and square both sides. This will clear the radical from our problem.

Example 32.

$$\begin{array}{rl}
\text{The solutions are } \sqrt{3} \text{ and } -\sqrt{3} & \text{Write as "one" expression equal to } x \\
x = \pm\sqrt{3} & \text{Square both sides} \\
x^2 = 3 & \text{Make equal to zero} \\
\underline{-3-3} & \text{Subtract 3 from both sides} \\
x^2 - 3 = 0 & \text{Our Solution}
\end{array}$$

We may have to isolate the term with the square root (with plus or minus) by adding or subtracting. With these problems remember to square a binomial we use the formula $(a+b)^2 = a^2 + 2ab + b^2$

Example 33.

$$\begin{array}{rl}
\text{The solutions are } 2-5\sqrt{2} \text{ and } 2+5\sqrt{2} & \text{Write as "one" expression equal to } x \\
x = 2 \pm 5\sqrt{2} & \text{Isolate the square root term} \\
\underline{-2-2\quad} & \text{Subtract 2 from both sides} \\
x - 2 = \pm 5\sqrt{2} & \text{Square both sides} \\
x^2 - 4x + 4 = 25 \cdot 2 & \\
x^2 - 4x + 4 = 50 & \text{Make equal to zero} \\
\underline{-50-50} & \text{Subtract 50} \\
x^2 - 4x - 46 = 0 & \text{Our Solution}
\end{array}$$

If the solution is a fraction we will clear it just as before by multiplying by the denominator.

Example 34.

$$\begin{array}{rl}
\text{The solutions are } \dfrac{2+\sqrt{3}}{4} \text{ and } \dfrac{2-\sqrt{3}}{4} & \text{Write as "one" expresion equal to } x
\end{array}$$

$$
\begin{array}{rl}
x = \dfrac{2 \pm \sqrt{3}}{4} & \text{Clear fraction by multiplying by 4} \\
4x = 2 \pm \sqrt{3} & \text{Isolate the square root term} \\
\underline{-2 \quad -2} & \text{Subtract 2 from both sides} \\
4x - 2 = \pm\sqrt{3} & \text{Square both sides} \\
16x^2 - 16x + 4 = 3 & \text{Make equal to zero} \\
\underline{-3 \quad -3} & \text{Subtract 3} \\
16x^2 - 16x + 1 = 0 & \text{Our Solution}
\end{array}
$$

The process used for complex solutions is identical to the process used for radicals.

Example 35.

$$
\begin{array}{rl}
\text{The solutions are } 4 - 5i \text{ and } 4 + 5i & \text{Write as "one" expression equal to } x \\
x = 4 \pm 5i & \text{Isolate the } i \text{ term} \\
\underline{-4 \quad -4} & \text{Subtract 4 from both sides} \\
x - 4 = \pm 5i & \text{Square both sides} \\
x^2 - 8x + 16 = 25i^2 & i^2 = -1 \\
x^2 - 8x + 16 = -25 & \text{Make equal to zero} \\
\underline{+25 \quad +25} & \text{Add 25 to both sides} \\
x^2 - 8x + 41 = 0 & \text{Our Solution}
\end{array}
$$

Example 36.

$$
\begin{array}{rl}
\text{The solutions are } \dfrac{3 - 5i}{2} \text{ and } \dfrac{3 + 5i}{2} & \text{Write as "one" expression equal to } x \\
x = \dfrac{3 \pm 5i}{2} & \text{Clear fraction by multiplying by denominator} \\
2x = 3 \pm 5i & \text{Isolate the } i \text{ term} \\
\underline{-3 \quad -3} & \text{Subtract 3 from both sides} \\
2x - 3 = \pm 5i & \text{Square both sides} \\
4x^2 - 12x + 9 = 5i^2 & i^2 = -1 \\
4x^2 - 12x + 9 = -25 & \text{Make equal to zero} \\
\underline{+25 \quad +25} & \text{Add 25 to both sides} \\
4x^2 - 12x + 34 = 0 & \text{Our Solution}
\end{array}
$$

Practice - Quadratic from Roots

From each problem, find a quadratic equation with those numbers as its solutions.

1) 2, 5

2) 3, 6

3) 20, 2

4) 13, 1

5) 4, 4

6) 0, 9

7) 0, 0

8) $-2, -5$

9) $-4, 11$

10) $3, -1$

11) $\frac{3}{4}, \frac{1}{4}$

12) $\frac{5}{8}, \frac{5}{7}$

13) $\frac{1}{2}, \frac{1}{3}$

14) $\frac{1}{2}, \frac{2}{3}$

15) $\frac{3}{7}, 4$

16) $2, \frac{2}{9}$

17) $-\frac{1}{3}, \frac{5}{6}$

18) $\frac{5}{3}, -\frac{1}{2}$

19) $-6, \frac{1}{9}$

20) $-\frac{2}{5}, 0$

21) ± 5

22) ± 1

23) $\pm \frac{1}{5}$

24) $\pm \sqrt{7}$

25) $\pm \sqrt{11}$

26) $\pm 2\sqrt{3}$

27) $\pm \frac{\sqrt{3}}{4}$

28) $\pm 11i$

29) $\pm i\sqrt{13}$

30) $\pm 5i\sqrt{2}$

31) $2 \pm \sqrt{6}$

32) $-3 \pm \sqrt{2}$

33) $1 \pm 3i$

34) $-2 \pm 4i$

35) $6 \pm i\sqrt{3}$

36) $-9 \pm i\sqrt{5}$

37) $\frac{-1 \pm \sqrt{6}}{2}$

38) $\frac{2 \pm 5i}{3}$

39) $\frac{6 \pm i\sqrt{2}}{8}$

40) $\frac{-2 \pm i\sqrt{15}}{2}$

9.6

Quadratics - Quadratic Substitution

We have seen three different ways to solve quadratics: factoring, completing the square, and the quadratic formula. A quadratic is any equation of the form $0 = ax^2 + bx + c$, however, we can use the skills learned to solve quadratics to solve problems with higher (or sometimes lower) powers if the equation is in what is called quadratic form.

Quadratic Form: $\boldsymbol{0 = ax^m + bx^n + c}$ **where** $\boldsymbol{m = 2n}$

An equation is in quadratic form if one of the exponents on a variable is double the exponent on the same variable somewhere else in the equation. If this is the case we can create a new variable, set it equal to the variable with smallest exponent. When we substitute this into the equation we will have a quadratic equation we can solve.

Example 37.

$$\begin{array}{rl}
x^4 - 13x^2 + 36 = 0 & \text{Quadratic form, one exponent, 4, double the other, 2} \\
y = x^2 & \text{New variable equal to the variable with smaller expoent} \\
y^2 = x^4 & \text{Square both sides} \\
y^2 - 13y + 36 = 0 & \text{Substitute } y \text{ for } x^2 \text{ and } y^2 \text{ for } x^4 \\
(y-9)(y-4) = 0 & \text{Solve. We can solve this equation by factoring} \\
y - 9 = 0 \ \text{ or } \ y - 4 = 0 & \text{Set each factor equal to zero} \\
\underline{+9+9 \qquad\quad +4+4} & \text{Solve each equation} \\
y = 9 \ \text{ or } \ y = 4 & \text{Solutions for } y\text{, need } x\text{. We will use } y = x^2 \text{ equation} \\
9 = x^2 \ \text{ or } \ 4 = x^2 & \text{Substitute values for } y \\
\pm\sqrt{9} = \sqrt{x^2} \ \text{ or } \ \pm\sqrt{4} = \sqrt{x^2} & \text{Solve using the even root property, simplify roots} \\
x = \pm 3, \pm 2 & \text{Our Solutions}
\end{array}$$

When we have higher powers of our variable, we could end up with many more solutions. The previous equation had four unique solutions.

Example 38.

$$a^{-2} - a^{-1} - 6 = 0 \qquad \text{Quadratic form, one exponent, } -2\text{, is double the other, } -1$$

$$
\begin{array}{rl}
b=a^{-1} & \text{Make } a \text{ new variable equal to the variable with lowest exponent} \\
b^2=a^{-2} & \text{Square both sides} \\
b^2-b-6=0 & \text{Substitute } b^2 \text{ for } a^{-2} \text{ and } b \text{ for } a^{-1} \\
(b-3)(b+2)=0 & \text{Solve. We will solve by factoring} \\
b-3=0 \ \text{ or } \ b+2=0 & \text{Set each factor equal to zero} \\
\underline{+3+3 \qquad\quad -2-2} & \text{Solve each equation} \\
b=3 \ \text{ or } \ b=-2 & \text{Solutions for } b\text{, still need } a\text{, substitute into } b=a^{-1} \\
3=a^{-1} \ \text{ or } \ -2=a^{-1} & \text{Raise both sides to } -1 \text{ power} \\
3^{-1}=a \ \text{ or } \ (-2)^{-1}=a & \text{Simplify negative exponents} \\
a=\dfrac{1}{3},-\dfrac{1}{2} & \text{Our Solution}
\end{array}
$$

Just as with regular quadratics, these problems will not always have rational solutions. We also can have irrational or complex solutions to our equations.

Example 39.

$$
\begin{array}{rl}
2x^4+x^2=6 & \text{Make equation equal to zero} \\
\underline{-6-6} & \text{Subtract 6 from both sides} \\
2x^4+x^2-6=0 & \text{Quadratic form, one exponent, 4, double the other, 2} \\
y=x^2 & \text{New variable equal variable with smallest exponent} \\
y^2=x^4 & \text{Square both sides} \\
2y^2+y-6=0 & \text{Solve. We will factor this equation} \\
(2y-3)(y+2)=0 & \text{Set each factor equal to zero} \\
2y-3=0 \ \text{ or } \ y+2=0 & \text{Solve each equation} \\
\underline{+3+3} \qquad\quad \underline{-2-2} & \\
\dfrac{2y}{2}=\dfrac{3}{2} \ \text{ or } \ y=-2 & \\
y=\dfrac{3}{2} \ \text{ or } \ y=-2 & \text{We have } y\text{, still need } x\text{. Substitute into } y=x^2 \\
\dfrac{3}{2}=x^2 \ \text{ or } \ -2=x^2 & \text{Square root of each side} \\
\pm\sqrt{\dfrac{3}{2}}=\sqrt{x^2} \ \text{ or } \ \pm\sqrt{-2}=\sqrt{x^2} & \text{Simplify each root, rationalize denominator} \\
x=\dfrac{\pm\sqrt{6}}{2},\pm i\sqrt{2} & \text{Our Solution}
\end{array}
$$

When we create a new variable for our substitution, it won't always be equal to just another variable. We can make our substitution variable equal to an expression as shown in the next example.

Example 40.

$3(x-7)^2-2(x-7)+5=0$	Quadratic form
$y=x-7$	Define new variable
$y^2=(x-7)^2$	Square both sides
$3y^2-2y+5=0$	Substitute values into origional
$(3y-5)(y+1)=0$	Factor
$3y-5=0$ or $y+1=0$	Set each factor equal to zero
$\underline{+5+5}$ $\quad$ $\underline{-1-1}$	Solve each equation
$3y=5$ or $y=-1$	
$\overline{3}\ \overline{3}$	
$y=\frac{5}{3}$ or $y=-1$	We have y, we still need x.
$\frac{5}{3}=x-7$ or $-1=x-7$	Substitute into $y=x-7$
$\underline{+\frac{21}{3}\quad +7}$ $\quad$ $\underline{+7\quad +7}$	Add 7. Use common denominator as needed
$x=\frac{26}{3},6$	Our Solution

Example 41.

$(x^2-6x)^2=7(x^2-6x)-12$	Make equation equal zero
$-7(x^2-6x)+12-7(x^2-6x)+12$	Move all terms to left
$(x^2-6x)^2-7(x^2-6x)+12=0$	Quadratic form
$y=x^2-6x$	Make new variable
$y^2=(x^2-6x)^2$	Square both sides
$y^2-7y+12=0$	Substitute into origional equation
$(y-3)(y-4)=0$	Solve by factoring
$y-3=0$ or $y-4=0$	Set each factor equal to zero
$\underline{+3+3}$ $\quad$ $\underline{+4+4}$	Solve each equation
$y=3$ or $y=4$	We have y, still need x.
$3=x^2-6x$ or $4=x^3-6x$	Solve each equation, complete the square
$\left(\frac{1}{2}\cdot 6\right)^2=3^2=9$	Add 9 to both sides of each equation
$12=x^2-6x+9$ or $13=x^2-6x+9$	Factor
$12=(x-3)^2$ or $13=(x-3)^2$	Use even root property

$$\begin{array}{rl}
\pm\sqrt{12}=\sqrt{(x-3)^2} \quad \text{or} \quad \pm\sqrt{13}=\sqrt{(x-3)^2} & \text{Simplify roots}\\
\pm 2\sqrt{3}=x-3 \quad \text{or} \quad \pm\sqrt{13}=x-3 & \text{Add 3 to both sides}\\
\underline{+3 \qquad +3} \qquad\qquad \underline{+3 \qquad +3} & \\
x=3\pm 2\sqrt{3}, 3\pm\sqrt{13} & \text{Our Solution}
\end{array}$$

The higher the exponent, the more solution we could have. This is is illustrated in the following example, one with six solutions.

Example 42.

$$\begin{array}{rl}
x^6-9x^3+8=0 & \text{Quadratic form, one exponent, 6, double the other, 3}\\
y=x^3 & \text{New variable equal to variable with lowest exponent}\\
y^2=x^6 & \text{Square both sides}\\
y^2-9y+8=0 & \text{Substitute } y^2 \text{ for } x^6 \text{ and } y \text{ for } x^3\\
(y-1)(y-8)=0 & \text{Solve. We will solve by factoring.}\\
y-1=0 \quad \text{or} \quad y-8=0 & \text{Set each factor equal to zero}\\
\underline{+1+1} \qquad\quad \underline{+8+8} & \text{Solve each equation}\\
y=1 \quad \text{or} \quad y=8 & \text{Solutions for } y\text{, we need } x\text{. Substitute into } y=x^3\\
x^3=1 \quad \text{or} \quad x^3=8 & \text{Set each equation equal to zero}\\
\underline{-1-1} \qquad \underline{-8-8} & \\
x^3-1=0 \quad \text{or} \quad x^3-8=0 & \text{Factor each equation, difference of cubes}\\
(x-1)(x^2+x+1)=0 & \text{First equation factored. Set each factor equal to zero}\\
x-1=0 \quad \text{or} \quad x^2+x+1=0 & \text{First equation is easy to solve}\\
\underline{+1+1} \qquad\qquad\qquad\qquad & \\
x=1 & \text{First solution}\\
\dfrac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}=\dfrac{1\pm i\sqrt{3}}{2} & \text{Quadratic formula on second factor}\\
(x-2)(x^2+2x+4)=0 & \text{Factor the second difference of cubes}\\
x-2=0 \quad \text{or} \quad x^2+2x+4=0 & \text{Set each factor equal to zero.}\\
\underline{+2+2} \qquad\qquad\qquad\qquad & \text{First equation is easy to solve}\\
x=2 & \text{Our fourth solution}\\
\dfrac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}=-1\pm i\sqrt{3} & \text{Quadtratic formula on second factor}\\
x=1,2,\dfrac{1\pm i\sqrt{3}}{2},-1\pm i\sqrt{3} & \text{Our final six solutions}
\end{array}$$

Practice - Equations with Quadratics

Solve each of the following equations. Some equations will have complex roots.

1) $x^4-5x^2+4=0$

2) $y^4-9y^2+20=0$

3) $m^4-7m^2-8=0$

4) $y^4-29y^2+100=0$

5) $a^4-50a^2+49=0$

6) $b^4-10b^2+9=0$

7) $x^4-25x^2+144=0$

8) $y^4-40y^2+144=0$

9) $m^4-20m^2+64=0$

10) $x^6-35x^3+216=0$

11) $z^6-216=19z^3$

12) $y^4-2y^2=24$

13) $6z^4-z^2=12$

14) $x^{-2}-x^{-1}-12=0$

15) $x^{\frac{2}{3}}-35=2x^{\frac{1}{3}}$

16) $5y^{-2}-20=21y^{-1}$

17) $y^{-6}+7y^{-3}=8$

18) $x^4-7x^2+12=0$

19) $x^4-2x^2-3=0$

20) $x^4+7x^2+10=0$

21) $2x^4-5x^2+2=0$

22) $2x^4-x^2-3=0$

23) $x^4=9x^2+8=0$

24) $x^6-10x^3+16=0$

25) $8x^6-9x^3+1=0$

26) $8x^6+7x^3-1=0$

27) $x^6-17x^4+16=0$

28) $(x-1)^2-4(x-1)=5$

29) $(y+b)^2-4(y+b)=21$

30) $(x+1)^2+6(x+1)+9=0$

31) $(y+2)^2-6(y+2)=16$

32) $(m-1)^2-5(m-1)=14$

33) $(x-3)^2-2(x-3)=35$

34) $(a+1)^2+2(a-1)=15$

35) $(r-1)^2-8(r-1)=20$

36) $2(x-1)^2-(x-1)=3$

37) $3(y+1)^2-14(y+1)=5$

38) $(x^2-3)^2-2(x^2-3)=3$

39) $(3x^2-2x)^2+5=5(3x^2-2x)$

40) $(x^2+x+3)^2+15=8(x^2+x+3)$

41) $2(3x+1)^{\frac{2}{3}}-5(3x+1)^{\frac{1}{3}}=88$

42) $(x^2+x)^2-8(x^2+x)+12=0$

43) $(x^2+2x)^2-2(x^2+2x)=3$

44) $(2x^2+3x)^2=8(2x^2+3x)+9$

45) $(2x^2-x)^2-4(2x^2-x)+3=0$

46) $(3x^2-4x)^2=3(3x^2-4x)+4$

9.7

Quadratics - Rectangles

An application of solving quadratic equations comes from the formula for the area of a rectangle. The area of a rectangle can be calculated by multiplying the width by the length. To solve problems with rectangles we will first draw a picture to represent the problem and use the picture to set up our equation.

Example 43.

The length of a rectangle is 3 more than the width. If the area is 40 square inches, what are the dimensions?

$$\begin{array}{rl} \boxed{\quad 40 \quad} x & \text{We do not know the width, } x. \\ x+3 \quad & \text{Length is 4 more, or } x+4\text{, and area is 40.} \\ x(x+3)=40 & \text{Multiply length by width to get area} \\ x^2+3x=40 & \text{Distribute} \\ \underline{-40-40} & \text{Make equation equal zero} \\ x^2+3x-40=0 & \text{Factor} \\ (x-5)(x+8)=0 & \text{Set each factor equal to zero} \\ x-5=0 \text{ or } x+8=0 & \text{Solve each equation} \\ \underline{+5+5} \qquad \underline{-8-8} & \\ x=5 \text{ or } x=-8 & \text{Our } x \text{ is } a \text{ width, } can't \text{ be negative.} \\ (5)+3=8 & \text{Length is } x+3\text{, substitute 5 for } x \text{ to find length} \\ 5 \text{ in by } 8 \text{ in} & \text{Our Solution} \end{array}$$

The above rectangle problem is very simple as there is only one rectangle involved. When we compare two rectangles, we may have to get a bit more creative.

Example 44.

If each side of a square is increased by 6, the area is multiplied by 16. Find the side of the original square.

$$\begin{array}{cl} \boxed{x^2} x & \text{Square has all sides the same length} \\ x & \text{Area is found by multiplying length by width} \end{array}$$

$\boxed{16x^2}\ x+6$	Each side is increased by 6,
$x+6$	Area is 16 times original area
$(x+6)(x+6)=16x^2$	Multiply length by width to get area
$x^2+12x+36=16x^2$	FOIL
$\underline{-16x^2 \qquad\qquad -16x^2}$	Make equation equal zero
$-15x^2+12x+36=0$	Divide each term by -1, changes the signs
$15x^2-12x-36=0$	Solve using the quadratic formula
$x=\dfrac{12\pm\sqrt{(-12)^2-4(15)(-36)}}{2(15)}$	Evaluate
$x=\dfrac{16\pm\sqrt{2304}}{30}$	
$x=\dfrac{16\pm 48}{30}$	Can′*t* have *a* negative solution, we will only add
$x=\dfrac{60}{30}=2$	Our x is the original square
2	Our Solution

Example 45.

The length of a rectangle is 4 ft greater than the width. If each dimension is increased by 3, the new area will be 33 square feet larger. Find the dimensions of the original rectangle.

$\boxed{x(x+4)}\ x$	We don′*t* know width, x, length is 4 more, $x+4$
$x+4$	Area is found by multiplying length by width
$\boxed{x(x+4)+33}\ x+3$	Increase each side by 3. width becomes $x+3$, length $x+4+3=x+7$
$x+7$	Area is 33 more than original, $x(x+4)+33$
$(x+3)(x+7)=x(x+4)+33$	Set up equation, length times width is area
$x^2+10x+21=x^2+4x+33$	Subtract x^2 from both sides
$\underline{-x^2 \qquad\qquad -x^2}$	
$10x+21=4x+33$	Move variables to one side
$\underline{-4x \qquad -4x}$	Subtract $4x$ from each side
$6x+21=33$	Subtract 21 from both sides
$\underline{-21\ -21}$	
$6x=12$	Divide both sides by 6
$\overline{6}\quad\overline{6}$	
$x=2$	x is the width of the original

$$\begin{array}{rl}
(2)+4=6 & x+4 \text{ is the length. Substitue 2 to find} \\
2\,\text{ft by }6\,\text{ft} & \text{Our Solution}
\end{array}$$

From one rectangle we can find two equations. Perimeter is found by adding all the sides of a polygon together. A rectangle as two widths and two lengths, both the same size. So we can use the equation $P = 2l + 2w$ (twice the length plus twice the width).

Example 46.

The area of a rectangle is 168 cm^2. The perimeter of the same rectangle is 52 cm. What are the dimensions of the rectangle?

$$\begin{array}{rl}
\boxed{}\,x & \text{We don}'t\text{ know anything about length or width} \\
y & \text{Use two variables, } x \text{ and } y \\
xy=168 & \text{Length times width gives the area.} \\
2x+2y=52 & \text{Also use perimeter formula.} \\
\underline{-2x \qquad -2x} & \text{Solve by substitution, isolate } y \\
2y=-2x+52 & \text{Divide each term by 2} \\
\overline{2} \qquad \overline{2} \quad \overline{2} & \\
y=-x+26 & \text{Substitute into area equation} \\
x(-x+26)=168 & \text{Distribute} \\
-x^2+26x=168 & \text{Divide each term by } -1\text{, changing all the signs} \\
x^2-26x=-168 & \text{Solve by completing the square.} \\
\left(\frac{1}{2}\cdot 26\right)^2=13^2=169 & \text{Find number to complete the square: } \left(\frac{1}{2}\cdot b\right)^2 \\
x^2-26x+324=1 & \text{Add 169 to both sides} \\
(x-13)^2=1 & \text{Factor} \\
x-13=\pm 1 & \text{Square root both sides} \\
\underline{+13 \quad +13} & \\
x=13\pm 1 & \text{Evaluate} \\
x=14 \;\text{ or }\; 12 & \text{Two options for first side.} \\
y=-(14)+26=12 & \text{Substitute 14 into } y=-x+26 \\
y=-(12)+26=14 & \text{Substitute 12 into } y=-x+26 \\
 & \text{Both are the same rectangle, variables switched!} \\
12\,\text{cm by }14\,\text{cm} & \text{Our Solution}
\end{array}$$

Practice - Rectangles

1) In a landscape plan, a rectangular flowerbed is designed to be 4 meters longer than it is wide. If 60 square meters are needed for the plants in the bed, what should the dimensions of the rectangular bed be?

2) If the side of a square is increased by 5 the area is multiplied by 4. Find the side of the original square.

3) A rectangular lot is 20 yards longer than it is wide and its area is 2400 square yards. Find the dimensions of the lot.

4) The length of a room is 8 ft greater than its width. If each dimension is increased by 2 ft, the area will be increased by 60 sq. ft. Find the dimensions of the rooms.

5) The length of a rectangular lot is 4 rods greater than its width, and its area is 60 square rods. Find the dimensions of the lot.

6) The length of a rectangle is 15 ft greater than its width. If each dimension is decreased by 2 ft, the area will be decreased by 106 ft^2. Find the dimensions.

7) A rectangular piece of paper is twice as long as a square piece and 3 inches wider. The area of the rectangular piece is 108 in^2. Find the dimensions of the square piece.

8) A room is one yard longer than it is wide. At 75¢ per sq. yd. a covering for the floor costs $31.50. Find the dimensions of the floor.

9) The area of a rectangle is 48 ft^2 and its perimeter is 32 ft. Find its length and width.

10) The dimensions of a picture inside a frame of uniform width are 12 by 16 inches. If the whole area (picture and frame) is 288 in^2, what is the width of the frame?

11) A mirror 14 inches by 15 inches has a frame of uniform width. If the area of the frame equals that of the mirror, what is the width of the frame.

12) A lawn is 60 ft by 80 ft. How wide a strip must be cut around it when mowing the grass to have cut half of it.

13) A grass plot 9 yards long and 6 yards wide has a path of uniform width around it. If the area of the path is equal to the area of the plot, determine the width of the path.

15) A page is to have a margin of 1 inch, and is to contain 35 in^2 of painting. How large must the page be if teh length is to exceed the width by 2 inches?

16) A picture 10 inches long by 8 inches wide has a frame whose area is one half the area of the picture. What are the outside dimensions of the frame?

17) A rectangular wheat field is 80 rods long by 60 rods wide. A strip of uniform width is cut around the field, so that half the grain is left standing in the form of a rectangular plot. How wide is the strip that is cut?

18) A picture 8 inches by 12 inches is placed in a frame of uniform width. If the area of the frame equals the area of the picture find the width of the frame.

19) A rectangular field 225 ft by 120 ft has a ring of uniform width cut around the outside edge. The ring leaves 65% of the field uncut in the center. What is the width of the ring?

20) One Saturday morning George goes out to cut his lot that is 100 ft by 120 ft. He starts cutting around the outside boundary spiraling around towards the center. By noon he has cut 60% of the lawn. What is the width of the ring that he has cut?

21) A frame is 15 in by 25 in and is of uniform width. The inside of the frame leaves 75% of the total area available for the picture. What is the width of the frame?

22) A farmer has a field 180 ft by 240 ft. He wants to increase the area of the field by 50% by cultivating a band of uniform width around the outside. How wide a band should he cultivate?

23) The farmer in the previous problem has a neighber who has a field 325 ft by 420 ft. His neighbor wants to increase the size of his field by 20% by cultivating a band of uniform width around the outside of his lot. How wide a band should his neighbor cultivate?

24) A third farmer has a field that is 500 ft by 550 ft. He wants to increase his field by 20%. How wide a ring should he cultivate around the outside of his field?

25) Donna has a garden that is 30 ft by 36 ft. She wants to increase the size of the garden by 40%. How wide a ring around teh outside should she cultivate?

26) A picture is 12 in by 25 in and is surrounded by a frame of uniform width. The area of the fram is 30% of the area of the picture. How wide is the fram?

27) A landscape architect is designing a rectangular flowerbed to be borded with 28 plants taht are placed 1 meter apart. He needs an inner rectangular space in the center for plants that must be 1 meter from the border of the bed and that require 24 square meters for planting. What should the overall dimensions of the flowerbed be?

9.8

Quadratics - Teamwork

If it takes one person 4 hours to paint a room and another person 12 hours to paint the same room, working together they could paint the room even quicker, it turns out they would paint the room in 3 hours together. This can be reasoned by the following logic, if the first person paints the room in 4 hours, she paints $\frac{1}{4}$ of the room each hour. If the second person takes 12 hours to paint the room, he paints $\frac{1}{12}$ of the room each hour. So together, each hour they paint $\frac{1}{4} + \frac{1}{12}$ of the room. Using a common denominator of 12 gives: $\frac{3}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$. This means each hour, working together they complete $\frac{1}{3}$ of the room. If $\frac{1}{3}$ is completed each hour, it follows that it will take 3 hours to complete the entire room.

This pattern is used to solve teamwork problems. If the first person does a job in A, a second person does a job in B, and together they can do a job in T (total). We can use the team work equation.

$$\textbf{Teamwork Equation: } \boldsymbol{\frac{1}{A} + \frac{1}{B} = \frac{1}{T}}$$

Often these problems will involve fractions. Rather than thinking of the first fraction as $\frac{1}{A}$, it may be better to think of it as the reciprocal of A's time.

Example 47.

Adan can clean a room in 3 hours. If his sister Maria helps, they can clean it in $2\frac{2}{5}$ hours. How long will it take Maria to do the job alone?

$2\frac{2}{5} = \frac{12}{5}$	Together time, $2\frac{2}{5}$, needs to be converted to fraction
Adan: 3, Maria: x, Total: $\frac{5}{12}$	Clearly state times for each and total, using x for Maria
$\frac{1}{3} + \frac{1}{x} = \frac{5}{12}$	Using reciprocals, add the individual times gives total
$\frac{1(12x)}{3} + \frac{1(12x)}{x} = \frac{5(12x)}{12}$	Multiply each term by LCD of $12x$
$4x + 12 = 5x$	Reduce each fraction

$$\begin{array}{rl} \underline{-4x \qquad -4x} & \text{Move variables to one side, subtracting } 4x \\ 12 = x & \text{Our solution for } x \\ \text{It takes Maria 12 hours} & \text{Our Solution} \end{array}$$

Somtimes we only know how two people's times are related to eachother as in the next example.

Example 48.

Mike takes twice as long as Rachel to complete a project. Together they can complete the project in 10 hours. How long will it take each of them to complete the project alone?

$$\begin{array}{rl} \text{Mike: } 2x, \text{Rachel: } x, \text{Total: } 10 & \text{Clearly define variables. If Rachel is } x, \text{Mike is } 2x \\ \frac{1}{2x} + \frac{1}{x} = \frac{1}{10} & \text{Using reciprocals, add individule times equaling total} \\ \frac{1(10x)}{2x} + \frac{1(10x)}{x} = \frac{1(10x)}{10} & \text{Multiply each term by LCD, } 10x \\ 5 + 10 = x & \text{Combine like terms} \\ 15 = x & \text{We have our } x, \text{we said } x \text{ was Rachel}'s \text{ time} \\ 2(15) = 30 & \text{Mike is double Rachel, this gives Mike}'s \text{ time.} \\ \text{Mike: 30 hr, Rachel: 15 hr} & \text{Our Solution} \end{array}$$

With problems such as these we will often end up with a quadratic to solve.

Example 49.

Brittney can build a large shed in 10 days less than Cosmo can. If they built it together it would take them 12 days. How long would it take each of them working alone?

$$\begin{array}{rl} \text{Britney: } x - 10, \text{Cosmo: } x, \text{Total: } 12 & \text{If cosmo is } x, \text{Britney is } x - 10 \\ \frac{1}{x-10} + \frac{1}{x} = \frac{1}{12} & \text{Using reciprocals, make equation} \\ \frac{1(12x(x-10))}{x-10} + \frac{1(12x(x-10))}{x} = \frac{1(12x(x-10))}{12} & \text{Multiply by LCD: } 12x(x-10) \end{array}$$

$$\begin{array}{rl}
12x+12(x-10)=x(x-10) & \text{Reduce fraction}\\
12x+12x-120=x^2-10x & \text{Distribute}\\
24x-120=x^2-10x & \text{Combine like terms}\\
\underline{-24x+120 \quad -24x+120} & \text{Move all terms to one side}\\
0=x^2-34x+120 & \text{Factor}\\
0=(x-30)(x-4) & \text{Set each factor equal to zero}\\
x-30=0 \text{ or } x-4=0 & \text{Solve each equation}\\
\underline{+30+30} \qquad \underline{+4+4} & \\
x=30 \text{ or } x=4 & \text{This, } x\text{, was defined as Cosmo.}\\
30-10=20 \text{ or } 4-10=-6 & \text{Find Britney, can}'t \text{ have negative time}\\
\text{Britney: 20 days, Cosmo: 30 days} & \text{Our Solution}
\end{array}$$

In the previous example, when solving, one of the possible times ended up negative. We can't have a negative amount of time to build a shed, so this possibility is ignored for this problem. Also, as we were solving, we had to factor $x^2-34x+120$. This may have been difficult to factor. We could have also chosen to complete the square or use the quadratic formula to find our solutions.

It is important that units match as we solve problems. This means we may have to convert minutes into hours to match the other units given in the problem.

Example 50.

An electrician can complete a job in one hour less than his apprentice. Together they do the job in 1 hour and 12 minutes. How long would it take each of them working alone?

$$\begin{array}{rl}
1\text{ hr }12\text{ min}=1\dfrac{12}{60}\text{ hr} & \text{Change 1 hour 12 minutes to mixed number}\\
1\dfrac{12}{60}=1\dfrac{1}{5}=\dfrac{6}{5} & \text{Reduce and convert to fraction}\\
\text{Electrician: } x-1\text{, Apprentice: } x\text{, Total: }\dfrac{6}{5} & \text{Clearly define variables}\\
\dfrac{1}{x-1}+\dfrac{1}{x}=\dfrac{5}{6} & \text{Using reciprocals, make equation}\\
\dfrac{1(6x(x-1))}{x-1}+\dfrac{1(6x(x-1))}{x}=\dfrac{5(6x(x-1)}{6} & \text{Multiply each term by LCD } 6x(x-1)\\
6x+6(x-1)=5x(x-1) & \text{Reduce each fraction}
\end{array}$$

$$
\begin{array}{rl}
6x+6x-6=5x^2-5x & \text{Distribute}\\
12x-6=5x^2-5x & \text{Combine like terms}\\
\underline{-12x+6 \quad -12x+6} & \text{Move all terms to one side of equation}\\
0=5x^2-17x+6 & \text{Factor}\\
0=(5x-2)(x-3) & \text{Set each factor equal to zero}\\
5x-2=0 \ \text{ or } \ x-3=0 & \text{Solve each equation}\\
\underline{+2+2} \qquad\quad \underline{+3+3} & \\
5x=2 \ \text{ or } \ x=3 & \\
\overline{5} \quad \overline{5} \qquad\qquad\quad & \\
x=\frac{2}{5} \ \text{ or } \ x=3 & \text{Subtract 1 from each to find electrician}\\
\frac{2}{5}-1=\frac{-3}{5} \ \text{ or } \ 3-1=2 & \text{Ignore negative.}\\
\text{Electrician: 2 hr, Apprentice: 3 hours} & \text{Our Solution}
\end{array}
$$

Very similar to a teamwork problem is when the two involved parts are working against each other. A common example of this is a sink that is filled by a pipe and emptied by a drain. If they are working against eachother we need to make one of the values negative to show they oppose eachother. This is shown in the next example..

Example 51.

A sink can be filled by a pipe in 5 minutes but it takes 7 minutes to drain a full sink. If both th pipe and the drain are open, how long will it take to fill the sink?

$$
\begin{array}{rl}
\text{Sink: 5, Drain: 7, Total: } x & \text{Define varaibles, drain is negative}\\
\frac{1}{5}-\frac{1}{7}=\frac{1}{x} & \text{Using reciprocals to make equation,}\\
 & \text{Subtract because they are opposite}\\
\frac{1(35x)}{5}-\frac{1(35x)}{7}=\frac{1(35x)}{x} & \text{Multiply each term by LCD: } 35x\\
7x-5x=35 & \text{Reduce fractions}\\
2x=35 & \text{Combine like terms}\\
\overline{2} \quad\ \overline{2} & \text{Divide each term by 2}\\
x=12.5 & \text{Our answer for } x\\
\text{12.5 min or 12 min 30 sec} & \text{Our Solution}
\end{array}
$$

Practice - Teamwork

1) Bills father can paint a room in two hours less than Bill can paint it. Working together they can complete the job in two hours and 24 minutes. How much time would each require working alone?

2) Of two inlet pipes, the smaller pipe takes four hours loner than the larger pipe to fill a pool. When both pipes are open, the pool is filled in three hours and forty-five minutes. If only the larger pipe is open, how many hours are required to fill the pool?

3) Jack can wash and wax teh family car in one hour less than Bob can. The two working together can complete teh job in $1\,\frac{1}{5}$ hours. How much time would each require if they worked alone?

4) If A can do a piece of work alone in 6 days and B can do it alone in 4 days, how long will it take the two working together to complete the job?

5) Working alone it takes John 8 hours longer than Carlos to do a job. Working together they can do the job in 3 hours. How long will it take each to do the job working alone?

6) A can do a piece of work in 3 days, B in 4 days, and C in 5 days each working alone. How long will it take them to do it working together?

7) A can do a piece of work in 4 days and B can do it in half the time. How long will it take them to do the work together?

8) A cistern can be filled by one pipe in 20 minutes and by another in 30 minutes. How long will it take both pipes together to fill the tank?

9) If A can do a piece of work in 24 days and A and B together can do it in 6 days, how long would it take B to do the work alone?

10) A carpenter and his assistant can do a piece of work in $3\frac{3}{4}$ days. If the carpenter himself could do teh work alone in 5 days, how long would the assistant take to do the work alone?

11) If Sam can do a certain job in 3 days, while it takes Fred 6 days to do the same job, how long will it take them, working together, to complete the job?

12) Tim can finish a certain job in 10 hours. It take his wife JoAnn only 8 hours to do the same job. If they work together, how long will it take them to complete the job?

13) Two people working together can complete a job in 6 hours. If one of them works twice as fast as the other, how long would it take the faster person, working alone, to do the job?

14) If two peopel working together can do a job in 3 hours, how long will it take the slower person to do the same job if one of them is 3 times as fast as the other?

15) A water tank can be filled by an inlet pipe in 8 hours. It takes twice that long for the outlet pipe to empty the tank. How long will it take to fill the tank if both pipes are open?

16) A sink can be filled from the faucet in 5 minutes. It takes only 3 minutes to empty the sink when the drain is open. If the sink is full and both the faucet and the drain are open, how lon will it take to empty the sink?

17) It takes 10 hours to fill a pool with the inlet pipe. It can be emptied in 15 hrs with the outlet pipe. If the pool is half full to begin with, how long will it take to fill it from there if both pipes are open?

18) A sink is $\frac{1}{4}$ full when both the faucet and the drain are opened. The faucet alone can fill the sink in 6 minutes, while it takes 8 minutes to empty it with the drain. How long will it take to fill the remaining $\frac{3}{4}$ of the sink?

19) A sink has two faucets, one for hot water and one for cold water. The sink can be filled by a cold-water faucet in 3.5 minutes. If both faucets are open, the sink is filled in 2.1 minutes. How long does it take to fill the sink with just the hot-water faucet open?

20) A water tank is being filled by two inlet pipes. Pipe A can fill the tank in $4\frac{1}{2}$ hrs, while both pipes together can fill teh tank in 2 hours. How long does it take to fill the tank using only pipe B?

21) A tank can be emptied by any one of three taps. The first can empty the tank in 20 minutes whlie the second takes 32 minutes. If all three working together could empty the tank in $8\frac{8}{59}$ minutes, how long would the third take to empty the tank?

22) One pipe can fill a cistern in $1\frac{1}{2}$ hours while a second pipe can fill it in $2\frac{1}{3}$ hrs. Three pipes working together fill the cistern in 42 minutes. How long would it take the third pipe alone to fill the tank?

23) Sam takes 6 hours longer than Susan to wax a floor. Working together they can wax the floor in 4 hours. How long will it take each of them working alone to wax the floor?

24) It takes Robert 9 hours longer than Paul to rapair a transmission. If it takes them $2\frac{2}{5}$ hours to do the job if they work together, how long will it take each of them working alone?

25) It takes Sally $10\frac{1}{2}$ minutes longer than Patricia to clean kup their dorm room. If they work together they can clean it in 5 minutes. How long will it take each of them if they work alone?

26) A takes $7\frac{1}{2}$ minutes longer than B to do a job. Working together they can do the job in 9 minutes. How long does it take each working alone?

27) Secretary A takes 6 minutes longer than Secretary B to type 10 pages of manuscript. If they divide the job and work together it will take them $8\frac{3}{4}$ minutes to type 10 pages. How long will it take each working alone to type the 10 pages?

28) It takes John 24 minutes longer than Sally to mow the lawn. If they work together they can mow the lawn in 9 minutes. How long will it take each to mow the lawn if they work alone?

9.9

Quadratics - Simultaneous Products

When solving a system of equations where the variables are multiplied together we can use the same idea of substitution that we used with linear equations. When we do so we may end up with a quadratic equation to solve. When we used substitution we solved for a variable and substitute this expression into the other equation. If we have two products we will choose a variable to solve for first and divide both sides of the equations by that variable or the factor containing the variable. This will create a situation where substitution can easily be done.

Example 52.

$xy = 48$ $(x+3)(y-2) = 54$	To solve for x, divide first equation by x, second by $x+3$
$y = \dfrac{48}{x}$ and $y - 2 = \dfrac{54}{x+3}$	Substitute $\dfrac{48}{x}$ for y in the second equation
$\dfrac{48}{x} - 2 = \dfrac{54}{x+3}$	Multiply each term by LCD: $x(x+3)$
$\dfrac{48x(x+3)}{x} - 2x(x+3) = \dfrac{54x(x+3)}{x+3}$	Reduce each fraction
$48(x+3) - 2x(x+3) = 54x$	Distribute
$48x + 144 - 2x^2 - 6x = 54x$	Combine like terms
$-2x^2 + 42x + 144 = 54x$	Make equation equal zero
$\underline{-54x \qquad -54x}$	Subtract $54x$ from both sides
$-2x^2 - 12x + 144 = 0$	Divide each term by GCF of -2
$x^2 + 6x - 72 = 0$	Factor
$(x-6)(x+12) = 0$	Set each factor equal to zero
$x - 6 = 0$ or $x + 12 = 0$	Solve each equation
$\underline{+6+6} \qquad \underline{-12-12}$	
$x = 6$ or $x = -12$	Substitute each solution into $xy = 48$
$\dfrac{6y}{6} = \dfrac{48}{6}$ or $\dfrac{-12y}{-12} = \dfrac{48}{-12}$	Solve each equation
$y = 8$ or $y = -4$	Our solutions for y,
$(6,8)$ or $(-12,-4)$	Our Solutions as ordered pairs

These simultaneous product equations will also solve by the exact same pattern. We pick a variable to solve for, divide each side by that variable, or factor containing the variable. This will allow us to use substitution to create a rational expression we can use to solve. Quite often these problems will have two solutions.

Example 53.

$$\begin{aligned} xy &= -35 && \text{To solve for } x\text{, divide the first} \\ (x+6)(y-2) &= 5 && \text{equation by } x\text{, second by } x+6 \end{aligned}$$

$$y = \frac{-35}{x} \text{ and } y - 2 = \frac{5}{x+6} \qquad \text{Substitute } \frac{-35}{x} \text{ for } y \text{ in the second equation}$$

$$\frac{-35}{x} - 2 = \frac{5}{x+6} \qquad \text{Multiply each term by LCD: } x(x+6)$$

$$\frac{-35x(x+6)}{x} - 2x(x+6) = \frac{5x(x+6)}{x+6} \qquad \text{Reduce fractions}$$

$$\begin{aligned}
-35(x+6) - 2x(x+6) &= 5x && \text{Distribute} \\
-35x - 210 - 2x^2 - 12x &= 5x && \text{Combine like terms} \\
-2x^2 - 47x - 210 &= 5x && \text{Make equation equal zero} \\
\underline{-5x \qquad\quad} & \underline{\;-5x} && \\
-2x^2 - 52x - 210 &= 0 && \text{Divide each term by } -2 \\
x^2 + 26x + 105 &= 0 && \text{Factor} \\
(x+5)(x+21) &= 0 && \text{Set each factor equal to zero} \\
x + 5 = 0 \text{ or } x + 21 &= 0 && \text{Solve each equation} \\
\underline{-5 - 5} \qquad \underline{-21 - 21} & && \\
x = -5 \text{ or } x &= -21 && \text{Substitute each solution into } xy = -35 \\
-5y = -35 \text{ or } -21y &= -35 && \text{Solve each equation} \\
\overline{-5} \quad \overline{-5} \qquad \overline{-21} \quad & \overline{-21} && \\
y = 7 \text{ or } y &= \frac{5}{3} && \text{Our solutions for } y \\
(-5, 7) \text{ or } & \left(-21, \frac{5}{3}\right) && \text{Our Solutions as ordered pairs}
\end{aligned}$$

The processes used here will be used as we solve applications of quadratics including distance problems and revenue problems. These will be covered in another section.

Practice - Simultaneous Product Equations

Solve.

1) $xy = 72$
$(x+2)(y-4) = 128$

2) $xy = 180$
$(x-1)(y-\frac{1}{2}) = 205$

3) $xy = 150$
$(x-6)(y+1) = 64$

4) $xy = 120$
$(x+2)(y-3) = 120$

5) $xy = 45$
$(x+2)(y+1) = 70$

6) $xy = 65$
$(x-8)(y+2) = 35$

7) $xy = 90$
$(x-5)(y+1) = 120$

8) $xy = 48$
$(x-6)(y+3) = 60$

9) $xy = 16$
$(x+1)(y-4) = 16$

10) $xy = 60$
$(x+5)(y+3) = 150$

11) $xy = 45$
$(x-5)(y+3) = 160$

12) $xy = 80$
$(x-5)(y+5) = 45$

9.10

Quadratics - Revenue and Distance

A common application of quadratics come from revenue and distance problems. Both are set up almost identical to each other so they are both included together. Once they are set up, we will solve them in exactly the same way we solved the simultaneous product equations.

Revenue problems are problems where a person buys a certain number of items for a certain price per item. If we multiply the number of items by the price per item we will get the total paid. To help us organize our information we will use the following table for revenue problems

	Number	Price	Total
First			
Second			

The price column will be used for the individual prices, the total column is used for the total paid, which is calculated by multiplying the number by the price. Once we have the table filled out we will have our equations which we can solve. This is shown in the following examples.

Example 54.

A man buys several fish for \$56. After three fish die, he decides to sell the rest at a profit of \$5 per fish. His total profit was \$4. How many fish did he buy to begin with?

	Number	Price	Total
Buy	n	p	56
Sell			

Using our table, we don$'t$ know the number he bought, or at what price, so we use varibles n and p. Total price was \$56.

	Number	Price	Total
Buy	n	p	56
Sell	$n-3$	$p+5$	60

When he sold, he sold 3 less $(n-3)$, for \$5 more $(p+5)$. Total profit was \$4, combined with \$56 spent is \$60

$$np = 56 \quad \text{Find equatinos by multiplying number by price}$$

$$(n-3)(p+5) = 60 \quad \text{These are } a \text{ simultaneous product}$$

$$p = \frac{56}{n} \text{ and } p+5 = \frac{60}{n-3} \quad \text{Solving for number, divide by } n \text{ or } (n-3)$$

$$\frac{56}{n}+5=\frac{60}{n-3} \qquad \text{Substitute } \frac{56}{n} \text{ for } p \text{ in second equation}$$

$$\frac{56n(n-3)}{n}+5n(n-3)=\frac{60n(n-3)}{n-3} \qquad \text{Multiply each term by LCD: } n(n-3)$$

$$56(n-3)+5n(n-3)=60n \qquad \text{Reduce fractions}$$

$$56n-168+5n^2-15n=60n \qquad \text{Combine like terms}$$

$$5n^2+41n-168=60n \qquad \text{Move all terms to one side}$$

$$\underline{-60n \qquad -60n}$$

$$5n^2-19n-168=0 \qquad \text{Solve with quadratic formula}$$

$$n=\frac{19\pm\sqrt{(-19)^2-4(5)(-168)}}{2(5)} \qquad \text{Simplify}$$

$$n=\frac{19\pm\sqrt{3721}}{10}=\frac{19\pm 61}{10} \qquad \text{We don't want negative solutions, only do } +$$

$$n=\frac{80}{10}=8 \qquad \text{This is our } n$$

$$8 \text{ fish} \qquad \text{Our Solution}$$

Example 55.

A group of students together bought a couch for their dorm that cost \$96. However, 2 students failed to pay their share, so each student had to pay \$4 more. How many students were in the original group?

	Number	Price	Total
Deal	n	p	96
Paid			

\$96 was paid, but we don't know the number or the price agreed upon by each student.

	Number	Price	Total
Deal	n	p	96
Paid	$n-2$	$p+4$	96

There were 2 less that actually paid $(n-2)$ and they had to pay \$4 more $(p+4)$. The total here is still \$96.

$$np=96 \qquad \text{Equations are product of number and price}$$

$$(n-2)(p+4)=96 \qquad \text{This is } a \text{ simultaneous product}$$

$$p=\frac{96}{n} \text{ and } p+4=\frac{96}{n-2} \qquad \text{Solving for number, divide by } n \text{ and } n-2$$

$$\frac{96}{n}+4=\frac{96}{n-2} \qquad \text{Substitute } \frac{96}{n} \text{ for } p \text{ in the second equation}$$

$$\frac{96n(n-2)}{n}+4n(n-2)=\frac{96n(n-2)}{n-2} \qquad \text{Multiply each term by LCD: } n(n-2)$$

$$\begin{aligned}
96(n-2)+4n(n-2)&=96n && \text{Reduce fractions}\\
96n-192+4n^2-8n&=96n && \text{Distribute}\\
4n^2+88n-192&=96n && \text{Combine like terms}\\
\underline{-96n\qquad\quad -96n} & && \text{Set equation equal to zero}\\
4n^2-8n-192&=0 && \text{Solve by completing the square,}\\
\underline{+192+192} & && \text{Separate variables and constant}\\
4n^2-8n&=192 && \text{Divide each term by } a \text{ or } 4\\
\overline{\;4\;}\quad\overline{\;4\;}\quad\overline{\;4\;} & &&\\
n^2-2n&=48 && \text{Complete the square: } \left(b\cdot\frac{1}{2}\right)^2\\
\left(2\cdot\frac{1}{2}\right)^2&=1^2=1 && \text{Add to both sides of equation}\\
n^2-2n+1&=49 && \text{Factor}\\
(n-1)^2&=49 && \text{Square root of both sides}\\
n-1&=\pm 7 && \text{Add 1 to both sides}\\
n&=1\pm 7 && \text{We don}'t \text{ want } a \text{ negative solution}\\
n&=1+7=8 &&\\
8\text{ students} & && \text{Our Solution}
\end{aligned}$$

The above examples were solved by the quadratic formula and completing the square. For either of these we could have use either method or even factoring. Remember we have several options for solving quadratics. Use the one that seems easiest for the problem.

Distance problems work with the same ideas that the revenue problems work. The only difference is the variables are r and t (for rate and time), instead of n and p (for number and price). We already know that distance is calculated by multiplying rate by time. So for our distance problems our table becomes the following:

	rate	time	distance
First			
Second			

Using this table and the exact same patterns as the revenue problems is shown in the following example.

Example 56.

Greg went to a conference in a city 120 miles away. On the way back, due to road construction he had to drive 10 mph slower which resulted in the return trip taking 2 hours longer. How fast did he drive on the way to the conference?

	rate	time	distance
There	r	t	120
Back			

We do not know rate, r, or time, t he traveled on the way to the conference. But we do know the distance was 120 miles.

	rate	time	distance
There	r	t	120
Back	$r-10$	$t+2$	120

Coming back he drove 10 mph slower $(r-10)$ and took 2 hours longer $(t+2)$. The distance was still 120 miles.

$$rt = 120 \quad \text{Equations are product of rate and time}$$
$$(r-10)(t+2) = 120 \quad \text{We have simultaneous product equations}$$

$$t = \frac{120}{r} \text{ and } t+2 = \frac{120}{r-10} \quad \text{Solving for rate, divide by } r \text{ and } r-10$$

$$\frac{120}{r} + 2 = \frac{120}{r-10} \quad \text{Substitute } \frac{120}{r} \text{ for } t \text{ in the second equation}$$

$$\frac{120r(r-10)}{r} + 2r(r-10) = \frac{120r(r-10)}{r-10} \quad \text{Multiply each term by LCD: } r(r-10)$$

$$120(r-10) + 2r^2 - 20r = 120r \quad \text{Reduce each fraction}$$
$$120r - 1200 + 2r^2 - 20r = 120r \quad \text{Distribute}$$
$$2r^2 + 100r - 1200 = 120r \quad \text{Combine like terms}$$
$$\underline{-120r \qquad\qquad -120r} \quad \text{Make equatino equal to zero}$$
$$2r^2 - 20r - 1200 = 0 \quad \text{Divide each term by 2}$$
$$r^2 - 10r - 600 = 0 \quad \text{Factor}$$
$$(r-30)(r+20) = 0 \quad \text{Set each factor equal to zero}$$
$$r - 30 = 0 \text{ and } r + 20 = 0 \quad \text{Solve each equation}$$
$$\underline{+30 + 30} \qquad \underline{-20 - 20}$$
$$r = 30 \text{ and } r = -20 \quad \text{Can't have a negative rate}$$
$$30 \text{ mph} \quad \text{Our Solution}$$

Another type of simultaneous product distance problem is where a boat is traveling in a river with the current or against the current (or an airplane flying with the wind or against the wind). If a boat is traveling downstream, the current will push it or increase the rate by the speed of the current. If a boat is traveling upstream, the current will pull against it or decrease the rate by the speed of the current. This is demonstrated in the following example.

Example 57.

A man rows down stream for 30 miles then turns around and returns to his original location, the total trip took 8 hours. If the current flows at 2 miles per hour, how fast would the man row in still water?

8

	rate	time	distance
down		t	30
up		$8-t$	30

Write total time above time column
We know the distance up and down is 30. Put t for time downstream. Subtracting $8-t$ becomes time upsteam

	rate	time	distance
down	$r+2$	t	30
up	$r-2$	$8-t$	30

Downstream the current of 2mph pushes the boat $(r+2)$ and upstream the current pulls the boat $(r-2)$

$(r+2)t=30$	Multiply rate by time to get equations
$(r-2)(8-t)=30$	We have a simultaneous product
$t=\dfrac{30}{r+2}$ and $8-t=\dfrac{30}{r-2}$	Solving for rate, divide by $r+2$ or $r-2$
$8-\dfrac{30}{r+2}=\dfrac{30}{r-2}$	Substitute $\dfrac{30}{r+2}$ for t in second equation
$8(r+2)(r-2)-\dfrac{30(r+2)(r-2)}{r+2}=\dfrac{30(r+2)(r-2)}{r-2}$	Multiply each term by LCD: $(r+2)(r-2)$
$8(r+2)(r-2)-30(r-2)=30(r+2)$	Reduce fractions
$8r^2-32-30r+60=30r+60$	Multiply and distribute
$8r^2-30r+28=30r+60$	Make equatino equal zero
$\underline{-30r-60-30r-60}$	
$8r^2-60r-32=0$	Divide each term by 4
$2r^2-15r-8=0$	Factor
$(2r+1)(r-8)=0$	Set each factor equal to zero
$2r+1=0$ or $r-8=0$	Solve each equation
$\underline{-1-1}$ $\underline{+8+8}$	
$\dfrac{2r}{2}=\dfrac{-1}{2}$ or $r=8$	
$r=-\dfrac{1}{2}$ or $r=8$	Can$'t$ have a negative rate
8 mph	Our Solution

Practice - Revenue and Distance

1) A merchant bought some pieces of silk for \$900. Had he bought 3 pieces more for the same money, he would have paid \$15 less for each piece. Find the number of pieces purchased.

2) A number of men subscribed a certain amount to make up a deficit of \$100 but 5 men failed to pay and thus increased the share of the others by \$1 each. Find the amount that each man paid.

3) A merchant bought a number of barrels of apples for \$120. He kept two barrels and sold the remainder at a profit of \$2 per barrel making a total profit of \$34. How many barrels did he originally buy?

4) A dealer bought a number of sheep for \$440. After 5 had died he sold the remainder at a profit of \$2 each making a profit of \$60 for the sheep. How many sheep did he originally purchase?

5) A man bought a number of articles at equal cost for \$500. He sold all but two for \$540 at a profit of \$5 for each item. How many articles did he buy?

6) A clothier bought a job lot of suits for \$750. He sold all but 3 of them for \$864 making a profit of \$7 on each suit sold. How many suits did he buy?

7) A goup of boys bought a boat for \$450. Five boys failed to pay their share, hence each remaining boys were compelled to pay \$4.50 more. How many boys were in the original group and how much had each agreed to pay?

8) The total expenses of a camping party were \$72. If there had been 3 fewer persons in the party, it would have cost each person \$2 more than it did. How many people were in the party and how much did it cost each one?

9) A factory tests the road performance of new model cars by driving them at two different rates of speed for at least 100 kilometers at each rate. The speed rates range from 50 to 70 km/hr in the lower range and from 70 to 90 km/hr in the higher range. A driver plans to test a car on an available speedway by driving it for 120 kilometers at a speed in the lower range and then driving 120 kilometers at a rate that is 20 km/hr faster. At what rates should he drive if he plans to complete te test in $3\frac{1}{2}$ hours?

10) A train traveled 240 kilometers at a certain speed. When the engine was replaced by an improved model, the speed was increased by 20 km/hr and the travel time for the trip was decreased by 1 hour. What was the rate of each engine?

11) The rate of teh current in a stream is 3 km/hr. A man rowed upstream for 3 kilometers and then returned. The round trip required 1 hour and 20 minutes. How fast was he rowing?

12) A pilot flying at a constant rate against a headwind of 50 km/hr flew for 750 kilometers, then reversed direction and returned to his starting point. He completed the round trip in 8 hours. What was the speed of the plane?

13) Two drivers are testing teh same model car at speeds that differ by 20 km/hr. The one driving at the slower rate drives 70 kilometers down a speedway and returns by the same route. The one driving at the faster rate drives 76 kilometers down the speedway and returns by the same route. Both drivers leave at the same time, and the faster car returns $\frac{1}{2}$ hour earlier than the slower car. At what rates were the cars driven?

14) An athlete plans to row upstream a distance of 2 kilometers and then return to his startin point in a total time of 2 hours and 20 minutes. If the rate of ther current is 2 km/hr, how fast should he row?

15) An automobile goes to a place 72 miles distant and then returns, the round trip occupying 9 hours. His speed in returning is 12 miles per hour faster than his speed in going. Find the rate of speed in both going and returning.

16) An automobile made a trip of 120 miles and then returned, the round trip occupying 7 hours. Returning the rate was increased 10 miles an hour. Find the rate of each.

17) The rate of a stream is 3 miles an hour. If a crew rows downstream for a distance of 8 miles and then back again, the round trip occupying 5 hours, what is the rate of teh crew in still water?

18) The railroad distance between two towns is 240 miles. If the speed of a train were incrased 4 miles an hour, the trip would take 40 minutes less. What is the usual rate of the train?

19) By going 15 miles per hour faster, a train would have required 1 hour less to travel 180 miles. How fast did it travel?

20) Mr. Jones visits his grandmother who lives 100 miles away on a regular basis. Recently a new freeway has opend up and, although the freeway rout is 120 miles, he can drive 20 mph faster on average and takes 30 minutes less time to make the trip. What is Mr. Jones rate of both the old route and on the freeway?

21) If a train had traveled 5 miles an hour faster, it would have needed $1\frac{1}{2}$ hours less time to travel 150 miles. Find the rate of the train.

22) A traveler having 18 miles to go, calculates that his usual rate would make him one-half hour late for an appointment; he finds that in order to arrive on time he must travel at a rate one-half mile an hour faster. What is his usual rate?

9.11

Quadratics - Graphing

Just as we drew pictures of the solutions for lines or linear equations, we can draw a picture of solution to quadratics as well. One way we can do that is to make a table of values.

Example 58.

$y = x^2 - 4x + 3$ Make a table of values

x	y
0	
1	
2	
3	
4	

We will test 5 values to get an idea of shape

$y = (0)^2 + 4(0) + 3 = 0 - 0 + 3 = 3$ Plug 0 in for x and evaluate

$y = (1)^2 - 4(1) + 3 = 1 - 4 + 3 = 0$ Plug 1 in for x and evaluate

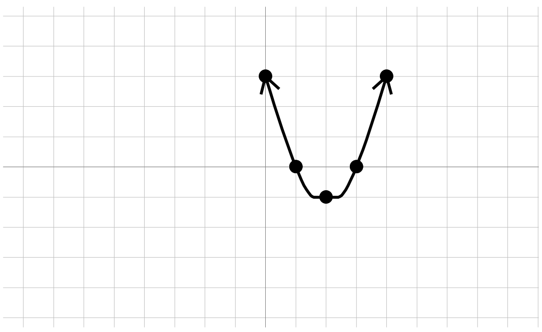
$y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 = -1$ Plug 2 in for x and evaluate

$y = (3)^2 - 4(3) + 3 = 9 - 12 + 3 = 0$ Plug 3 in for x and evaluate

$y = (4)^2 - 4(4) + 3 = 16 - 16 + 3 = 3$ Plug 4 in for x and evaluate

x	y
0	3
1	0
2	-1
3	0
4	3

Our completed table. Plot points on graph

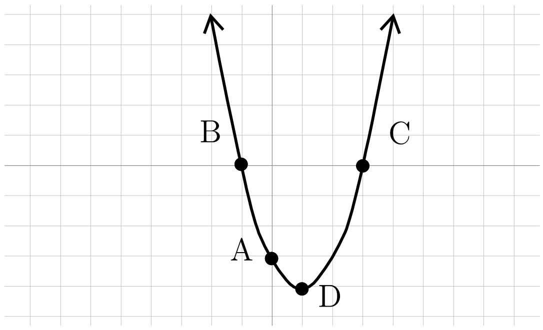


Plot the points $(0, 3)$, $(1, 0)$, $(2, -1)$, $(3, 0)$, and $(4, 3)$.

Connect the dots with a smooth curve.

Our Solution

When we have x^2 in our equations, the graph will no longer be a straight line. Quadratics have a graph that looks like a U shape that is called a parabola. The above method to graph a parabola works for any equation, however, it can be very tedious to find all the correct points to get the correct bend and shape. For this reason we identify several key points on a graph and in the equation to help us graph parabolas more efficiently. These key points are described below.



Point A: y-intercept: Where the graph crosses the virticle y-axis.

Points B and C: x-intercepts: Where the graph crosses the horizontal x-axis

Point D: Vertex: The point where the graph curves and changes directions.

We will use the following method to find each of the points on our parabola.

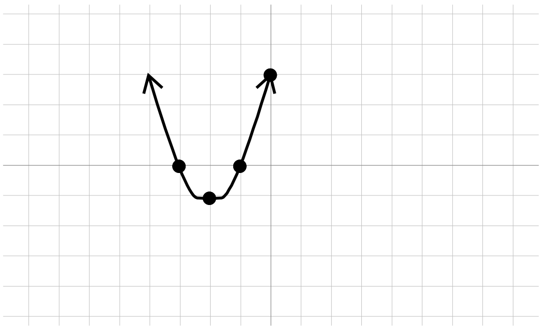
To graph the equation $y = ax^2 + bx + c$, find the following poinst

1. y-intercept: Found by makign $x = 0$, this simplifies down to $y = c$
2. x-intercepts: Found by making $y = 0$, this means solving $0 = ax^2 + bx + c$
3. Vertex: Let $x = \frac{-b}{2a}$ to find x. Then plug this value into equation to find y

After finding these points we can connect the dots with a smooth curve to find our graph!

Example 59.

$$\begin{aligned}
y = x^2 + 4x + 3 &\quad \text{Find the key points} \\
\\
y = 3 &\quad y = c \,\text{is the}\, y - \text{intercept} \\
\\
0 = x^2 + 4x + 3 &\quad \text{To find}\, x - \text{intercept we solve the equation} \\
0 = (x + 3)(x + 1) &\quad \text{Factor} \\
x + 3 = 0 \;\text{ and }\; x + 1 = 0 &\quad \text{Set each factor equal to zero} \\
\underline{-3 - 3} \qquad\qquad \underline{-1 - 1} &\quad \text{Solve each equation} \\
x = -3 \;\text{ and }\; x = -1 &\quad \text{Our}\, x - \text{intercepts} \\
\\
x = \frac{-4}{2(1)} = \frac{-4}{2} = -2 &\quad \text{To find the vertex, first use}\, x = \frac{-b}{2a} \\
y = (-2)^2 + 4(-2) + 3 &\quad \text{Plug this answer into equation to find}\, y - \text{coordinate} \\
y = 4 - 8 + 3 &\quad \text{Evaluate} \\
y = -1 &\quad \text{The}\, y - \text{coordinate} \\
(-2, -1) &\quad \text{Vertex as}\, a\, \text{point}
\end{aligned}$$



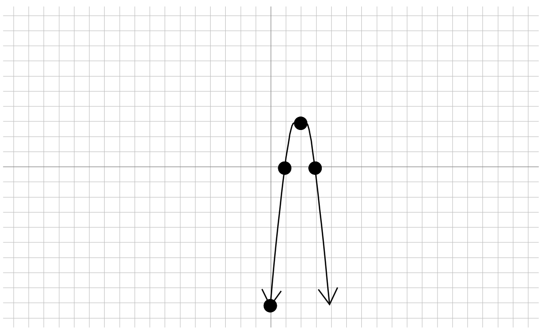
Graph the y-intercept at 3, the x-intercepts at -3 and -1, and the vertex at $(-2,-1)$. Connect the dots with smooth curve in a U shape to get our parabola.

Our Solution

If the a in $y=ax^2+bx+c$ is a negative value, the parabola will end up being an upside-down U. The process to graph it is identical, we just need to be very careful of how our signs operate. Remember, if a is negative, then ax^2 will also be negative because we only square the x, not the a.

Example 60.

$$\begin{array}{rl}
y=-3x^2+12x-9 & \text{Find key points} \\
\\
y=-9 & y-\text{intercept is } y=c \\
\\
0=-3x^2+12x-9 & \text{To find } x-\text{intercept solve this equation} \\
0=-3(x^2-4x+3) & \text{Factor out GCF first, then factor rest} \\
0=-3(x-3)(x-1) & \text{Set each factor with } a \text{ varaible equal to zero} \\
x-3=0 \text{ and } x-1=0 & \text{Solve each equation} \\
\underline{+3+3} \qquad\quad \underline{+1+1} & \\
x=3 \text{ and } x=1 & \text{Our } x-\text{intercepts} \\
\\
x=\dfrac{-12}{2(-3)}=\dfrac{-12}{-6}=2 & \text{To find the vertex, first use } x=\dfrac{-b}{2a} \\
y=-3(2)^2+12(2)-9 & \text{Plug this value into equation to find } y-\text{coordinate} \\
y=-3(4)+24-9 & \text{Evaluate} \\
y=-12+24-9 & \\
y=3 & y-\text{value of vertex} \\
(2,3) & \text{Vertex as } a \text{ point}
\end{array}$$



Graph the y-intercept at -9, the x-intercepts at 3 and 1, and the vertex at (2, 3). Connect the dots with smooth curve in an upside-down U shape to get our parabola.

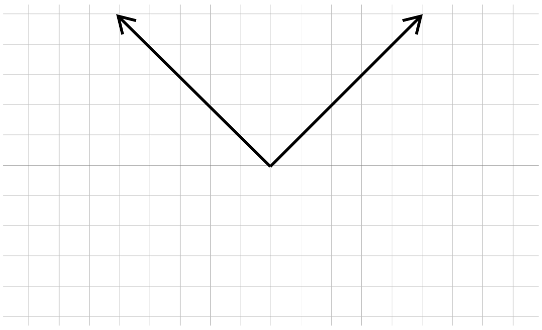
Our Solution

It is important to remember the graph of all quadratics is a parabolas with the same U shape (they could be upside-down). If you plot your poits and we can't connect them in the correct U shape then one of your points must be wrong. Go back and check the your work to be sure they are correct!

Just as all quadratics (equation with $y = x^2$) all have the same U-shape to them and all linear equations (equations such as $y = x$) have the same line shape when graphed, different equations have different shapes to them. Below are some common equations (some we have yet to cover!) with their graph shape drawn.

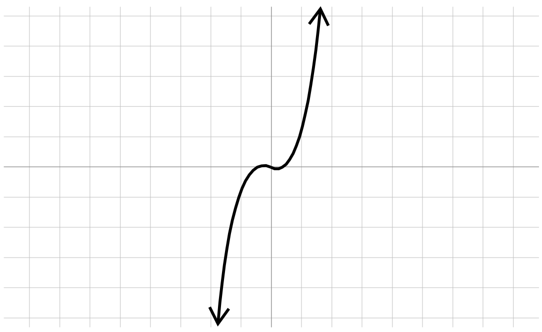
Absolute Value

$y = |x|$



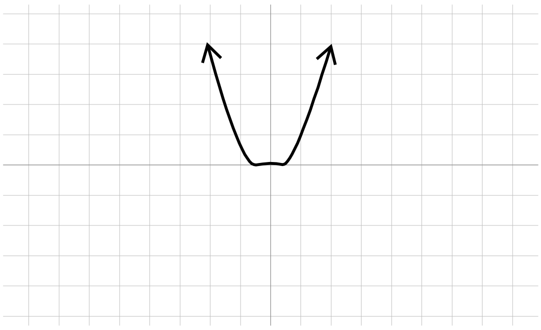
Cubic

$y = x^3$



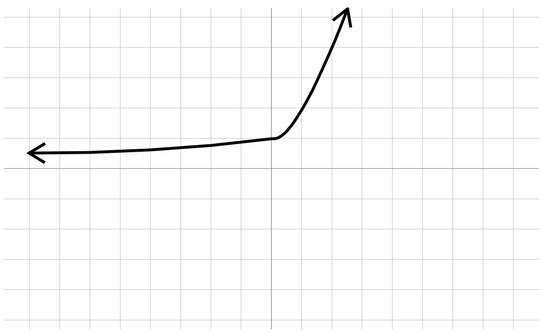
Quadratic

$y = x^2$



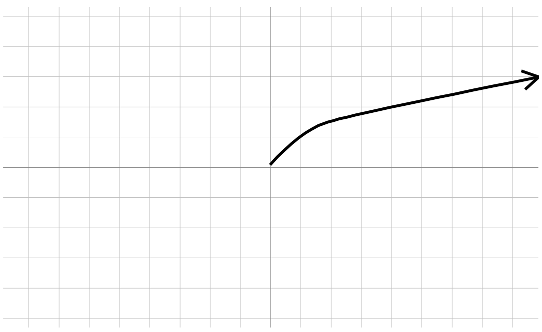
Exponential

$y = a^x$



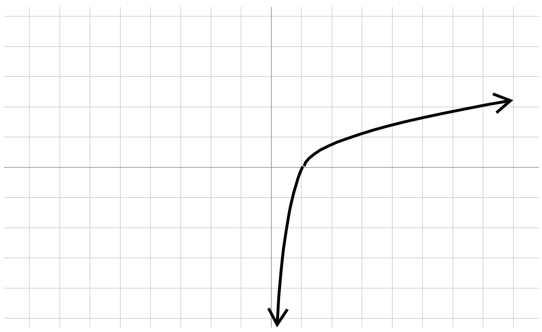
Square Root

$y = \sqrt{x}$



Logarithmic

$y = \log_a x$



Practice - Graphing Quadratic Functions

Find the vertex and intercepts of the following quadratics. Use this information to graph the quadratic.

1) $y = x^2 - 2x - 8$

2) $y = x^2 - 2x - 3$

3) $y = 2x^2 - 12x + 10$

4) $y = 2x^2 - 12x + 16$

5) $y = -2x^2 + 12x - 18$

6) $y = -2x^2 + 12x - 10$

7) $y = -3x^2 + 24x - 45$

8) $y = -3x^2 + 12x - 9$

9) $y = -x^2 + 4x + 55$

10) $y = -x^2 + 4x - 3$

11) $y = -x^2 + 6x - 5$

12) $y = -2x^2 + 16x - 30$

13) $y = -2x^2 + 16x - 24$

14) $y = 2x^2 + 4x - 6$

15) $y = 3x^2 + 12x + 9$

16) $y = 5x^2 + 30x + 45$

17) $y = 5x^2 - 40x + 75$

18) $y = 5x^2 + 20x + 15$

19) $y = -5x^2 - 60x - 175$

20) $y = -5x^2 + 20x - 15$

Answers - Solving with Radicals

1) 3
2) 3
3) 1, 5
4) no solution
5) ± 2
6) 3
7) $\frac{1}{4}$
8) no solution
9) 5
10) 7
11) 6
12) 46
13) 5
14) 21
15) $-\frac{3}{2}$
16) $-\frac{7}{3}$

Answers - Solving with Exponents

1) $\pm 5\sqrt{3}$
2) -2
3) $\pm 2\sqrt{2}$
4) 3
5) $\pm 2\sqrt{6}$
6) $-3, 11$
7) -5
8) $\frac{1}{5}, -\frac{3}{5}$
9) -1
10) $\frac{-1 \pm 3\sqrt{2}}{2}$
11) 65, -63
12) 5
13) -7
14) $-\frac{11}{2}, \frac{5}{2}$
15) $\frac{11}{2}, \frac{5}{2}$
16) $-\frac{191}{64}$
17) $-\frac{3}{8}, -\frac{5}{8}$
18) $\frac{9}{8}$
19) $\frac{5}{4}$
20) No Solution
21) $-\frac{34}{3}, -3$
22) 3
23) $-\frac{17}{2}$
24) No Solutoin

Answers - Complete the Square

1) 225; $(x-15)^2$
2) 144; $(a-12)^2$
3) 324; $(m-18)^2$
4) 289; $(x-17)^2$
5) $\frac{225}{4}; (x-\frac{15}{2})^2$
6) $\frac{1}{324}; (r-\frac{1}{18})^2$
7) $\frac{1}{4}; (y-\frac{1}{2})^2$
8) $\frac{289}{4}; (p-\frac{17}{2})^2$
9) $\{11, 5\}$
10) $\{4+2\sqrt{7}, 4-2\sqrt{7}\}$
11) $\{4+i\sqrt{29}, 4-i\sqrt{29}\}$
12) $\{-1+I\sqrt{42}, -1-I\sqrt{42}\}$
13) $\{\frac{-2+I\sqrt{38}}{2}, \frac{-2-I\sqrt{38}}{2}\}$
14) $\{\frac{3+2i\sqrt{33}}{3}, \frac{3-3i\sqrt{33}}{3}\}$
15) $\{\frac{5+i\sqrt{215}}{5}, \frac{5-i\sqrt{215}}{5}\}$
16) $\{\frac{-4+3\sqrt{2}}{4}, \frac{-4-3\sqrt{2}}{4}\}$
17) $\{-5+\sqrt{86}, -5-\sqrt{86}\}$
18) $\{8+2\sqrt{30}, 8-2\sqrt{30}\}$
19) $\{9, 7\}$
20) $\{9, -1\}$
21) $\{-1+i\sqrt{21}, -1-i\sqrt{21}\}$
22) $\{1, -3\}$
23) $\{\frac{3}{2}, -\frac{7}{2}\}$
24) $\{3, -1\}$
25) $\{-5+2i, -5-2i\}$
26) $\{7+\sqrt{85}, 7-\sqrt{85}\}$

27) $\{7, 3\}$

28) $\{4, -14\}$

29) $\{1 + i\sqrt{2}, 1 - i\sqrt{2}\}$

30) $\{\frac{5 + i\sqrt{105}}{5}, \frac{5 - i\sqrt{105}}{5}\}$

31) $\{\frac{4 + i\sqrt{110}}{2}, \frac{4 - i\sqrt{110}}{2}\}$

32) $\{1, -3\}$

33) $\{4 + i\sqrt{39}, 4 - i\sqrt{39}\}$

34) $\{-1. -7\}$

35) $\{7, 1\}$

36) $\{2, -6\}$

37) $\{\frac{-6 + i\sqrt{258}}{6}, \frac{-6 - i\sqrt{258}}{6}\}$

38) $\{\frac{-6 + i\sqrt{111}}{3}, \frac{-6 - i\sqrt{111}}{3}\}$

39) $\{\frac{5 + i\sqrt{130}}{5}, \frac{5 - i\sqrt{130}}{5}\}$

40) $\{2, -4\}$

41) $\{\frac{-5 + i\sqrt{87}}{2}, \frac{-5 - i\sqrt{87}}{2}\}$

42) $\{\frac{-7 + \sqrt{181}}{2}, \frac{-7 - \sqrt{181}}{2}\}$

43) $\{\frac{3 + i\sqrt{271}}{7}, \frac{3 - i\sqrt{271}}{7}\}$

44) $\{\frac{-1 + 2i\sqrt{6}}{2}, \frac{-1 - 2i\sqrt{6}}{2}\}$

45) $\{\frac{7 + i\sqrt{139}}{2}, \frac{7 - i\sqrt{139}}{2}\}$

46) $\{\frac{5 + i\sqrt{67}}{2}, \frac{5 - i\sqrt{67}}{2}\}$

47) $\{\frac{12}{5}, -4\}$

48) $\{\frac{1 + i\sqrt{511}}{4}, \frac{1 - i\sqrt{511}}{4}\}$

49) $\{\frac{9 + \sqrt{21}}{2}, \frac{9 - \sqrt{21}}{2}\}$

50) $\{\frac{1 + i\sqrt{163}}{2}, \frac{1 - i163}{2}\}$

51) $\{\frac{-5 + i\sqrt{415}}{8}, \frac{-5 - i\sqrt{415}}{8}\}$

52) $\{\frac{11 + i\sqrt{95}}{6}, \frac{11 - i\sqrt{95}}{6}\}$

53) $\{\frac{5 + i\sqrt{191}}{2}, \frac{5 - i\sqrt{191}}{2}\}$

54) $\{\frac{15 + i\sqrt{3}}{2}, \frac{15 - i\sqrt{3}}{2}\}$

55) $\{1, -\frac{5}{2}\}$

56) $\{3, -\frac{3}{2}\}$

Answers - Quadratic Formula

1) $\{\frac{i\sqrt{6}}{2}, -\frac{i\sqrt{6}}{2}\}$

2) $\{\frac{i\sqrt{6}}{3}, -\frac{i\sqrt{6}}{3}\}$

3) $\{2 + \sqrt{5}, 2 - \sqrt{5}\}$

4) $\{\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}\}$

5) $\{\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}\}$

6) $\{\frac{-1 + i\sqrt{29}}{5}, \frac{-1 - i\sqrt{29}}{5}\}$

7) $\{1, -\frac{1}{3}\}$

8) $\{\frac{1 + \sqrt{31}}{2}, \frac{1 - \sqrt{31}}{2}\}$

9) $\{3, -3\}$

10) $\{i\sqrt{2}, -i\sqrt{2}\}$

11) $\{3, 1\}$

12) $\{-1 + i, -1 - i\}$

13) $\{\frac{-3 + i\sqrt{55}}{4}, \frac{-3 - i\sqrt{55}}{4}\}$

14) $\{\frac{-3 + i\sqrt{159}}{12}, \frac{-3 - i\sqrt{159}}{12}\}$

15) $\{\frac{-3 + \sqrt{141}}{6}, \frac{-3 - \sqrt{141}}{6}\}$

16) $\{\sqrt{3}, -\sqrt{3}\}$

17) $\{\frac{-3 + \sqrt{401}}{14}, \frac{-3 - \sqrt{401}}{14}\}$

18) $\{\frac{-5 + \sqrt{137}}{8}, \frac{-5 - \sqrt{137}}{8}\}$

19) $\{2, -5\}$

20) $\{5, -9\}$

21) $\{\frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}\}$

22) $\{3, -\frac{1}{3}\}$

23) $\{\frac{7}{2}, -7\}$

24) $\{\frac{-3+i\sqrt{3}}{3}, \frac{-3-i\sqrt{3}}{3}\}$

25) $\{\frac{7+3\sqrt{21}}{10}, \frac{7-3\sqrt{21}}{10}\}$

26) $\{\frac{-5+\sqrt{337}}{12}, \frac{-5-\sqrt{337}}{12}\}$

27) $\{\frac{-3+i\sqrt{247}}{16}, \frac{-3-i\sqrt{247}}{16}\}$

28) $\{\frac{3+\sqrt{33}}{6}, \frac{3-\sqrt{33}}{6}\}$

29) $\{-1, -\frac{3}{2}\}$

30) $\{2\sqrt{2}, -2\sqrt{2}\}$

31) $\{4, -4\}$

32) $\{2, -4\}$

33) $\{4, -9\}$

34) $\{\frac{2+3i\sqrt{5}}{7}, \frac{2-3i\sqrt{5}}{7}\}$

35) $\{6, -\frac{9}{2}\}$

36) $\{\frac{5+i\sqrt{143}}{14}, \frac{5-i\sqrt{143}}{14}\}$

37) $\{\frac{-3+\sqrt{345}}{14}, \frac{-3-\sqrt{345}}{14}\}$

38) $\{\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{2}\}$

39) $\{\frac{\sqrt{26}}{2}, -\frac{\sqrt{26}}{2}\}$

40) $\{\frac{-1+\sqrt{141}}{10}, \frac{-1-\sqrt{141}}{10}\}$

Answers - Quadratics from Roots

NOTE: There are multiple answers for each problem. Try checking your answers because your answer may also be correct.

1) $x^2-7x+10$

2) $x^2-9x+18=0$

3) $x^2-22x+40=0$

4) $x^2-14x+13=0$

5) $x^2-8x+16=0$

6) $x^2-9x=0$

7) $x^2=0$

8) $x^2+7x+10=0$

9) $x^2-7x-44=0$

10) $x^2-2x-3=0$

11) $16x^2-16x+3=0$

12) $56x^2-75x-40=0$

13) $6x^2-7x+2=0$

14) $6x^2-7x+2=0$

15) $7x^2-40x+12=0$

16) $9x^2-2x+4=0$

17) $18x^2-9x-5=0$

18) $6x^2-7x-5=0$

19) $9x^2+53x-6=0$

20) $5x^2+2x=0$

21) $x^2-25=0$

22) $x^2-1=0$

23) $25x^2-1=0$

24) $x^2-7=0$

25) $x^2-11=0$

26) $x^2-12=0$

27) $16x^2-3=0$

28) $x^2+121=0$

29) $x^2+13=0$

30) $x^2+50=0$

31) $x^2-4x-2=0$

32) $x^2+6x+7=0$

33) $x^2-2x+10=0$

34) $x^2+4x+20=0$

35) $x^2-12x+39=0$

36) $x^2+18x+86=0$

37) $4x^2+4x-5=0$

38) $9x^2-6x+29=0$

39) $64x^2-96x+38=0$

40) $4x^2+8x+19=0$

Answers - Equations with Quadratics

1) $\pm 1, \pm 2$

2) $\pm 2, \pm\sqrt{5}$

3) $\pm i, \pm 2\sqrt{2}$

4) $\pm 5, \pm 2$

5) $\pm 1, \pm 7$

6) $\pm 3, \pm 1$

7) $\pm 3, \pm 4$

8) $\pm 6, \pm 2$

9) $\pm 2, \pm 4$

10) $2, 3, -1 \pm i\sqrt{3}, \frac{-3 \pm i\sqrt{3}}{2}$

11) $-2, 3, 1 \pm i\sqrt{3}, \frac{-3 \pm i\sqrt{3}}{2}$

12) $\pm\sqrt{6}, \pm 2i$

13) $\frac{\pm 2i\sqrt{3}}{3}, \frac{\pm\sqrt{6}}{2}$

14) $\frac{1}{4}, -\frac{1}{3}$

15) $-125, 343$

16) $-\frac{5}{4}, \frac{1}{5}$

17) $1, -\frac{1}{2}, \frac{1 \pm i\sqrt{3}}{4}, \frac{1 \pm i\sqrt{3}}{2}$

18) $\pm 2, \pm\sqrt{3}$

19) $\pm i, \pm\sqrt{3}$

20) $\pm\sqrt{5}, \pm\sqrt{2}$

21) $\pm\sqrt{2}, \pm\frac{\sqrt{2}}{2}$

22) $\pm i, \frac{\pm 6}{2}$

23) $\pm 1, \pm 2\sqrt{2}$

24) $2, \sqrt[3]{2}, -1 \pm i\sqrt{3}, \frac{-\sqrt[3]{2} \pm i\sqrt[6]{108}}{2}$

25) $1, \frac{1}{2}, \frac{-1 \pm i\sqrt{3}}{4}, \frac{-1 \pm i\sqrt{3}}{2}$

26) $\frac{1}{2}, -1, \frac{-1 \pm i\sqrt{3}}{4}, \frac{1 \pm i\sqrt{3}}{2}$

27) $\pm 1, \pm i, \pm 2, \pm 2i$

28) 4, 0

29) $-(b+3), 7-b$

30) -4

31) $-4, 6$

32) $8, -1$

33) $-2, 10$

34) 2, -6

35) $-1, 11$

36) $\frac{5}{2}, 0$

37) 4, $-\frac{4}{3}$

38) $\pm\sqrt{6}, \pm\sqrt{2}$

39) $\pm 1, \pm i, \pm 2, \pm 2i$

40) $0, \pm 1, -2$

41) $\frac{511}{3}, -\frac{1339}{24}$

42) $-3, \pm 2, 1$

43) $\pm 1, -3$

44) $-3, -1, \frac{3}{2}, -\frac{1}{2}$

45) $\pm 1, -\frac{1}{2}, \frac{3}{2}$

46) $1, 2, \frac{1}{3}, -\frac{2}{3}$

Answers - Rectangles

1) 6 m x 10 m

2) 5

3) 40 yd x 60 yd

4) 10 ft x 18 ft

5) 6 x 10

6) 20 ft x 35 ft

7) 6” x 6”

8) 6 yd x 7 yd

9) 4 ft x 12 ft

10) 1.54 in

11) 3 in

12) 10 ft

13) 1.5 yd

14) 7 x 9

15) 1 in

16) 10 rods

17) 2 in

18) 15 ft

19) 60 ft

20) 20 ft

21) 1.25 in

22) 23.16 ft

23) 17.5 ft

24) 25 ft

25) 3 ft

26) 1.145 in

27) 6 m x 8 m

Answers - Teamwork

1) 4 and 6
2) 6 hours
3) 2 and 3
4) 2.4
5) C = 4, J = 12
6) 1.28 days
7) $1\frac{1}{3}$ days
8) 12 min
9) 8 days
10) 15 days
11) 2 days
12) $4\frac{4}{9}$ days
13) 9 hours
14) 12 hours
15) 16 hours
16) $7\frac{1}{2}$ min
17) 15 hours
18) 18 min
19) $5\frac{1}{4}$ min
20) 3.6 hours
21) 24 min
22) 180 min or 3 hrs
23) Su = 6, Sa = 12
24) 3 hrs and 12 hrs
25) P = 7, S = $17\frac{1}{2}$
26) 15 and 22.5 min
27) A = 21, B = 15
28) 12 and 36 min

Answers - Simultaneous Product Equations

1) $(2,36),(-18,-4)$
2) $(-9,-20),(-40,-\frac{9}{5})$
3) $(10,15),(-90,-\frac{5}{3})$
4) $(8,15),(-10,-12)$
5) $(5,9),(18,2.5)$
6) $(13,5),(-20,-\frac{13}{4})$
7) $(45,2),(-10,-9)$
8) $(16,3),(-6,-8)$
9) $(1,12),(-3,-4)$
10) $(20,3),(5,12)$
11) $(45,1),(-\frac{5}{3},-27)$
12) $(8,10),(-10,-8)$

Answers - Revenue and Distance

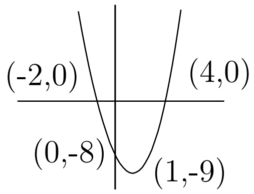
1) 12
2) \$5
3) 24
4) 55
5) 20
6) 30
7) 25 @ \$18
8) 12 @ \$6
9) 60 mph, 80 mph
10) 60, 80
11) 6 km/hr
12) 200 km/hr 48.
13) 56, 76
14) 3.033 km/hr
15) 12 mph, 24 mph
16) 30 mph, 40 mph
17) r = 5
18) 36 mph
19) 45 mph
20) 40 mph, 60 mph

21) 20 mph

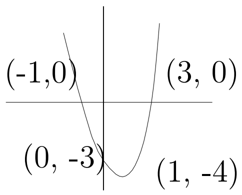
22) 4 mph

Answers - Graphing Quadratic Functions

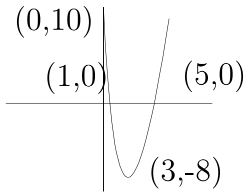
1)



2)



3)



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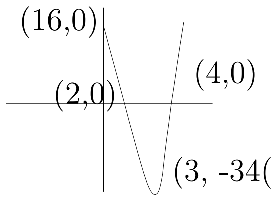

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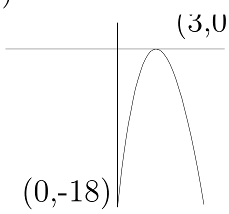

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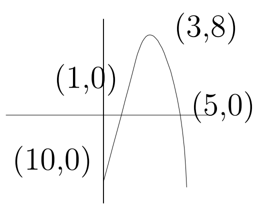

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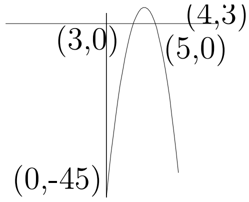

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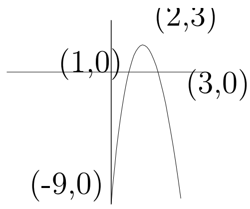

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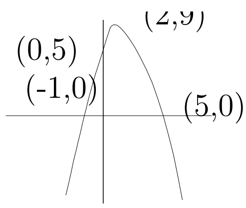

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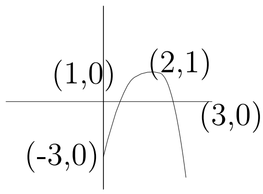

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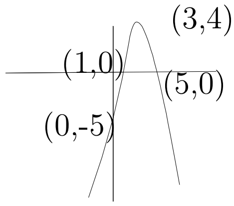

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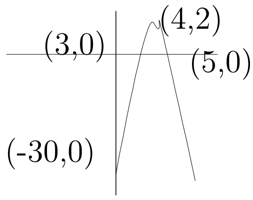

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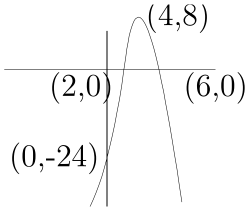

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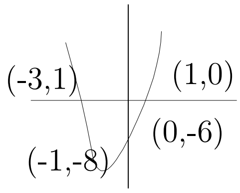

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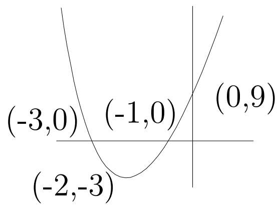

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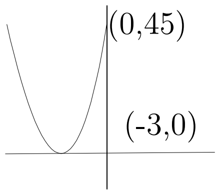

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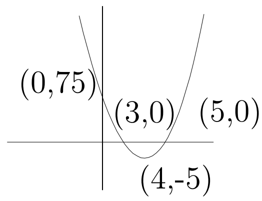

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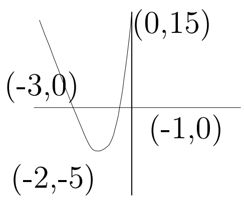

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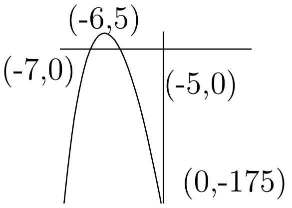

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