Beginning and Intermediate Algebra

Chapter 8: Radicals

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Chapter 8: Radicals

8.1

Radicals - Square Roots

Square roots are the most common type of radical used. A square root "un-squares" a number. For example, because $5^2 = 25$ we say the square root of 25 is 5. The square root of 25 is written as $\sqrt{25}$. The following example gives several square roots:

Example 1.

$\sqrt{1} = 1$	$\sqrt{121} = 11$
$\sqrt{4} = 2$	$\sqrt{625} = 25$
$\sqrt{9} = 3$	$\sqrt{-81} = \text{Undefined}$

The final example, $\sqrt{-81}$ is currently undefined as negatives have no square root. This is because if we square a positive or a negative, the answer will be positive. Thus we can only take square roots of positive numbers. In another lesson we will define a method we can use to work with and evaluate negative square roots, but for now we will simply say they are undefined.

Not all numbers have a nice even square root. For example, if we found $\sqrt{8}$ on our calculator, the answer would be 2.828427124746190097603377448419... and even this number is a rounded approximation of the square root. To be as accurate as possible, we will never use the calculator to find decimal approximations of square roots. Instead we will express roots in simplest radical form. We will do this using a property known as the product rule of radicals

$$\textbf{Product Rule of Square Roots: } \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

We can use the product rule to simplify an expression such as $\sqrt{36 \cdot 5}$ by spliting it into two roots, $\sqrt{36}\cdot\sqrt{5}$, and simplifying the first root, $6\sqrt{5}$. The trick in this process is being able to translate a problem like $\sqrt{180}$ into $\sqrt{36\cdot 5}$. There are several ways this can be done. The most common and, with a bit of practice, the fastest method, is to find perfect squares that divide evenly into the radicand, or number under the radical. This is shown in the next example.

Example 2.

$$\begin{array}{rl}
\sqrt{75} & \text{75 is divisible by 25, } a \text{ perfect square} \\
\sqrt{25\cdot 3} & \text{Split into factors} \\
\sqrt{25}\cdot\sqrt{3} & \text{Product rule, take the square root of 25} \\
5\sqrt{3} & \text{Our Solution}
\end{array}$$

If there is a coefficient in front of the radical to begin with, the problem mearly becomes a big multiplication problem.

Example 3.

$$\begin{array}{rl}
5\sqrt{63} & \text{63 is divisible by 9, } a \text{ perfect square} \\
5\sqrt{9\cdot 7} & \text{Split into factors} \\
5\sqrt{9}\cdot\sqrt{7} & \text{Product rule, take the square root of 9} \\
5\cdot 3\sqrt{7} & \text{Multiply coefficients} \\
15\sqrt{7} & \text{Our Solution}
\end{array}$$

As we simplify radicals using this method it is important to be sure our final answer can be simplifyied no more.

Example 4.

$$\begin{array}{rl}
\sqrt{72} & \text{72 is divisible by 9, } a \text{ perfect square} \\
\sqrt{9\cdot 8} & \text{Split into factors} \\
\sqrt{9}\cdot\sqrt{8} & \text{Product rule, take the square root of 9} \\
3\sqrt{8} & \text{But 8 is also divisible by } a \text{ perfect square, 4} \\
3\sqrt{4\cdot 2} & \text{Split into factors} \\
3\sqrt{4}\cdot\sqrt{2} & \text{Product rule, take the square root of 4}
\end{array}$$

$$3 \cdot 2\sqrt{2} \quad \text{Multiply}$$
$$6\sqrt{2} \quad \text{Our Solution.}$$

The previous example could have been done in fewer steps if we had noticed that $72 = 36 \cdot 2$, but often the time it takes to discover the larger perfect square is more than it would take to simplify in several steps.

Variables often are part of the radicand as well. When taking the square roots of variables, we can divide the exponent by 2. For example, $\sqrt{x^8} = x^4$, because we divide the exponent of 8 by 2. This follows from the power of a power rule of expoennts, $(x^4)^2 = x^8$. When squaring, we multiply the exponent by two, so when taking a square root we divide the exponent by 2. This is shown in the following example.

Example 5.

$$-5\sqrt{18x^4y^6z^{10}} \quad \text{18 is divisible by 9, } a \text{ perfect square}$$
$$-5\sqrt{9 \cdot 2x^4y^6z^{10}} \quad \text{Split into factors}$$
$$-5\sqrt{9} \cdot \sqrt{2} \cdot \sqrt{x^4} \cdot \sqrt{y^6} \cdot \sqrt{z^{10}} \quad \text{Product rule, simplify roots, divide exponents by 2}$$
$$-5 \cdot 3x^2y^3z^5\sqrt{2} \quad \text{Multiply coefficients}$$
$$-15x^2y^3z^5\sqrt{2} \quad \text{Our Solution}$$

We can't always evenly divide the exponent on a variable by 2. Sometimes we have a remainder. If there is a remainder, this means the remainder is left inside the radical, and the whole number part is how many are outside the radical. This is shown in the following example.

Example 6.

$$\sqrt{20x^5y^9z^6} \quad \text{20 is divisible by 4, } a \text{ perfect square}$$
$$\sqrt{4 \cdot 5x^5y^9z^6} \quad \text{Split into factors}$$
$$\sqrt{4} \cdot \sqrt{5} \cdot \sqrt{x^5} \cdot \sqrt{y^9} \cdot \sqrt{z^6} \quad \text{Simplfy, divide exponents by 2, remainder is left inside}$$
$$2x^2y^4z^3\sqrt{5xy} \quad \text{Our Solution}$$

Practice - Square Roots

Simplify.

1) $\sqrt{245}$

2) $\sqrt{125}$

3) $\sqrt{36}$

4) $\sqrt{196}$

5) $\sqrt{12}$

6) $\sqrt{72}$

7) $3\sqrt{12}$

8) $5\sqrt{32}$

9) $6\sqrt{128}$

10) $7\sqrt{128}$

11) $-8\sqrt{392}$

12) $-7\sqrt{63}$

13) $\sqrt{192n}$

14) $\sqrt{343b}$

15) $\sqrt{196v^2}$

16) $\sqrt{100n^3}$

17) $\sqrt{252x^2}$

18) $\sqrt{200a^3}$

19) $-\sqrt{100k^4}$

20) $-4\sqrt{175p^4}$

21) $-7\sqrt{64x^4}$

22) $-2\sqrt{128n}$

23) $-5\sqrt{36m}$

24) $8\sqrt{112p^2}$

25) $\sqrt{45x^2y^2}$

26) $\sqrt{72a^3b^4}$

27) $\sqrt{16x^3y^3}$

28) $\sqrt{512a^4b^2}$

29) $\sqrt{320x^4y^4}$

30) $\sqrt{512m^4n^3}$

31) $3\sqrt{320x^4y^4}$

32) $8\sqrt{98mn}$

33) $5\sqrt{245x^2y^3}$

34) $2\sqrt{72x^2y^2}$

35) $-2\sqrt{180u^3v}$

36) $-5\sqrt{72x^3y^4}$

37) $-8\sqrt{180x^4y^2z^4}$

38) $6\sqrt{50a^4bc^2}$

39) $2\sqrt{80hj^4k}$

40) $-\sqrt{32xy^2z^3}$

41) $-4\sqrt{54mnp^2}$

42) $-8\sqrt{32m^2p^4q}$

8.2

Radicals - Higher Roots

While square roots are the most common type of radical we work with, we can take higher roots of numbers as well: cube roots, fourth roots, fifth roots, etc. Following is a definition of radicals.

$$\boldsymbol{\sqrt[m]{a} = b \text{ if } b^m = a}$$

The small letter m inside the radical is called the index. It tells us which root we are taking, or which power we are "un-doing". For square roots the index is 2. As this is the most common root, the two is not usually written. The following example includes several higher roots.

Example 7.

$\sqrt[3]{125} = 5$	$\sqrt[3]{-64} = -4$
$\sqrt[4]{81} = 3$	$\sqrt[7]{-128} = -2$
$\sqrt[5]{32} = 2$	$\sqrt[4]{-16} = \text{undefined}$

We must be careful of a few things as we work with higher roots. First its important not to forget to check the index on the root. $\sqrt{81} = 9$ but $\sqrt[4]{81} = 3$. This is because $9^2 = 81$ and $3^4 = 81$. Another thing to watch out for is negatives under roots. We can take an odd root of a negative number, because a negative number raised to an odd power is still negative. However, we cannot take an even root of a negative number, this we will say is undefined. In a later section we will discuss how to work with roots of negative, but for now we will simply say they are undefined.

We can simplify higher roots in much the same way we simplified square roots, using the product property of radicals.

$$\textbf{Product Property of Radicals: } \boldsymbol{\sqrt[m]{ab} = \sqrt[m]{a} \cdot \sqrt[m]{b}}$$

Often we are not as familiar with higher powers as we are with squares. It is important to remember what index we are working with as we try and work our way to the solution.

Example 8.

$\sqrt[3]{54}$	We are working with a cubed root, want third powers
$2^3 = 8$	Test 2, $2^3 = 8$, 54 is not divisible by 8.
$3^3 = 27$	Test 3, $3^3 = 27$, 54 is divisible by 57!
$\sqrt[3]{27 \cdot 2}$	Write as factors
$\sqrt[3]{27} \cdot \sqrt[3]{2}$	Product rule, take cubed root of 27
$3\sqrt[3]{2}$	Our Solution

Just as with square roots, if we have a coefficient, we multiply the new coefficients together.

Example 9.

$3\sqrt[4]{48}$	We are working with a fourth root, want fourth powers
$2^4 = 16$	Test $2, 2^4 = 16$, 48 is divisible by 16!
$3\sqrt[4]{16 \cdot 3}$	Write as factors
$3\sqrt[4]{16} \cdot \sqrt[4]{3}$	Product rule, take fourth root of 16
$3 \cdot 2\sqrt[4]{3}$	Multiply coefficients
$6\sqrt[4]{3}$	Our Solution

We can also take higher roots of variables. As we do, we will divide the exponent on the variable by the index. Any whole answer is how many of that varible will come out of the square root. Any remainder is how many are left behind inside the square root. This is shown in the following examples.

Example 10.

$\sqrt[5]{x^{25}y^{17}z^3}$	Divide each exponent by 5, whole number outside, remainder inside
$x^5y^3\sqrt[5]{y^2z^3}$	Our Solution

In the previous example, for the x, we divided $\frac{25}{5} = 5\,R\,0$, so x^5 came out, no x's remain inside. For the y, we divided $\frac{17}{5} = 3R2$, so y^3 came out, and y^2 remains inside. For the z, when we divided $\frac{3}{5} = 0R3$, all three or z^3 remained inside. The following example includes integers in our problem.

Example 11.

$2\sqrt[3]{40a^4b^8}$	Looking for cubes that divide into 40. The number 8 works!
$2\sqrt[3]{8 \cdot 5a^4b^8}$	Take cube root of 8, dividing exponents on variables by 3
$2 \cdot 2ab^2\sqrt[3]{5ab^2}$	Remainders are left in radical. Multiply coefficients
$4ab^2\sqrt[3]{5ab^2}$	Our Solution

Practice - Higher Roots

Simplify.

1) $\sqrt[3]{625}$

2) $\sqrt[3]{375}$

3) $\sqrt[3]{750}$

4) $\sqrt[3]{250}$

5) $\sqrt[3]{875}$

6) $\sqrt[3]{24}$

7) $-4\sqrt[4]{96}$

8) $-8\sqrt[4]{48}$

9) $6\sqrt[4]{112}$

10) $3\sqrt[4]{48}$

11) $-\sqrt[4]{112}$

12) $5\sqrt[4]{243}$

13) $\sqrt[4]{648a^2}$

14) $\sqrt[4]{64n^3}$

15) $\sqrt[5]{224n^3}$

16) $\sqrt[5]{-96x^3}$

17) $\sqrt[5]{224p^5}$

18) $\sqrt[6]{256x^6}$

19) $-3\sqrt[7]{896r}$

20) $-8\sqrt[7]{384b^8}$

21) $-2\sqrt[3]{-48v^7}$

22) $4\sqrt[3]{250a^6}$

23) $-7\sqrt[3]{320n^6}$

24) $-\sqrt[3]{512n^6}$

25) $\sqrt[3]{-135x^5y^3}$

26) $\sqrt[3]{64u^5v^3}$

27) $\sqrt[3]{-32x^4y^4}$

28) $\sqrt[3]{1000a^4b^5}$

29) $\sqrt[3]{256x^4y^6}$

30) $\sqrt[3]{189x^3y^6}$

31) $7\sqrt[3]{-81x^3y^7}$

32) $-4\sqrt[3]{56x^2y^8}$

33) $2\sqrt[3]{375u^2v^8}$

34) $8\sqrt[3]{-750xy}$

35) $-3\sqrt[3]{192ab^2}$

36) $3\sqrt[3]{135xy^3}$

37) $6\sqrt[3]{-54m^8n^3p^7}$

38) $-6\sqrt[4]{80m^4p^7q^4}$

39) $6\sqrt[4]{648x^5y^7z^2}$

40) $-6\sqrt[4]{405a^5b^8c}$

41) $7\sqrt[4]{128h^6j^8k^8}$

42) $-6\sqrt[4]{324x^7yz^7}$

8.3

Radicals - Add/Subtract Radicals

Adding and subtracting radicals is very similar to adding and subtracting with variables. Consider the following example.

Example 12.

$$5x+3x-2x \quad \text{Combine like terms}$$
$$6x \quad \text{Our Solution}$$

$$5\sqrt{11}+3\sqrt{11}-2\sqrt{11} \quad \text{Combine like terms}$$
$$6\sqrt{11} \quad \text{Our Solution}$$

Notice that when we combined the terms with $\sqrt{11}$ it was just like combining terms with x. When adding and subtracting with radicals we can combine like radicals just like like terms. We add and subtract the coefficients in front of the radical, and the radical stays the same. This is shown in the following example.

Example 13.

$$
\begin{array}{rl}
7\sqrt[5]{6}+4\sqrt[5]{3}-9\sqrt[5]{3}+\sqrt[5]{6} & \text{Combine like radicals } 7\sqrt[5]{6}+\sqrt[5]{6} \text{ and } 4\sqrt[5]{3}-8\sqrt[5]{3} \\
8\sqrt[5]{6}-5\sqrt[5]{3} & \text{Our Solution}
\end{array}
$$

We cannot simplify this expression any more as the radicals do not match. Often problems we solve have no like radicals, however, if we simplify the radicals first we may find we do in fact have like radicals.

Example 14.

$$
\begin{array}{rl}
5\sqrt{45}+6\sqrt{18}-2\sqrt{98}+\sqrt{20} & \text{Simplify radicals, find perfect square factors} \\
5\sqrt{9\cdot 5}+6\sqrt{9\cdot 2}-2\sqrt{49\cdot 2}+\sqrt{4\cdot 5} & \text{Take roots where possible} \\
5\cdot 3\sqrt{5}+6\cdot 3\sqrt{2}-2\cdot 7\sqrt{2}+2\sqrt{5} & \text{Multiply coefficients} \\
15\sqrt{5}+18\sqrt{2}-14\sqrt{2}+2\sqrt{5} & \text{Combine like terms} \\
17\sqrt{5}+4\sqrt{2} & \text{Our Solution}
\end{array}
$$

This exact process can be used to add and subtract radicals with higher indicies

Example 15.

$$
\begin{array}{rl}
4\sqrt[3]{54}-9\sqrt[3]{16}+5\sqrt[3]{9} & \text{Simplify each radical, finding perfect cube factors} \\
4\sqrt[3]{27\cdot 2}-9\sqrt[3]{8\cdot 2}+5\sqrt[3]{9} & \text{Take roots where possible} \\
4\cdot 3\sqrt[3]{2}-9\cdot 2\sqrt[3]{2}+5\sqrt[3]{9} & \text{Multiply coefficients} \\
12\sqrt[3]{2}-18\sqrt[3]{2}+5\sqrt[3]{9} & \text{Combien like terms } 12\sqrt[3]{2}-18\sqrt[3]{2} \\
-6\sqrt[3]{2}+5\sqrt[3]{9} & \text{Our Solution}
\end{array}
$$

Practice - Add and Subtract Radicals

Simiplify

1) $2\sqrt{5}+2\sqrt{5}+2\sqrt{5}$

2) $-3\sqrt{6}-3\sqrt{3}-2\sqrt{3}$

3) $-3\sqrt{2}+3\sqrt{5}+3\sqrt{5}$

4) $-2\sqrt{6}-\sqrt{3}-3\sqrt{6}$

5) $-2\sqrt{6}-2\sqrt{6}-\sqrt{6}$

6) $-3\sqrt{3}+2\sqrt{3}-2\sqrt{3}$

7) $3\sqrt{6}+3\sqrt{5}+2\sqrt{5}$

8) $-\sqrt{5}+2\sqrt{3}-2\sqrt{3}$

9) $2\sqrt{2}-3\sqrt{18}-\sqrt{2}$

10) $-\sqrt{54}-3\sqrt{6}+3\sqrt{27}$

11) $-3\sqrt{6}-\sqrt{12}+3\sqrt{3}$

12) $-\sqrt{5}-\sqrt{5}-2\sqrt{54}$

13) $3\sqrt{2}+2\sqrt{8}-3\sqrt{18}$

14) $2\sqrt{20}+2\sqrt{20}-\sqrt{3}$

15) $3\sqrt{18}-\sqrt{2}-3\sqrt{2}$

16) $-3\sqrt{27}+2\sqrt{3}-\sqrt{12}$

17) $-3\sqrt{6}-3\sqrt{6}-\sqrt{3}+3\sqrt{6}$

18) $-2\sqrt{2}-\sqrt{2}+3\sqrt{8}+3\sqrt{6}$

19) $-2\sqrt{18}-3\sqrt{8}-\sqrt{20}+2\sqrt{20}$

20) $-3\sqrt{18}-\sqrt{8}+2\sqrt{8}+2\sqrt{8}$

21) $-2\sqrt{24}-2\sqrt{6}+2\sqrt{6}+2\sqrt{20}$

22) $-3\sqrt{8}-\sqrt{5}-3\sqrt{6}+2\sqrt{18}$

23) $3\sqrt{24}-3\sqrt{27}+2\sqrt{6}+2\sqrt{8}$

24) $2\sqrt{6}-\sqrt{54}-3\sqrt{27}-\sqrt{3}$

25) $-2\sqrt[3]{16}+2\sqrt[3]{16}-2\sqrt[3]{2}$

26) $3\sqrt[3]{135}-\sqrt[3]{81}-\sqrt[3]{135}$

27) $2\sqrt[4]{243}-2\sqrt[4]{243}-\sqrt[4]{3}$

28) $-3\sqrt[4]{4}+3\sqrt[4]{324}+2\sqrt[4]{64}$

29) $3\sqrt[4]{2}-2\sqrt[4]{2}-\sqrt[4]{243}$

30) $2\sqrt[4]{6}+2\sqrt[4]{4}+3\sqrt[4]{6}$

31) $-\sqrt[4]{324}+3\sqrt[4]{324}-3\sqrt[4]{4}$

32) $-2\sqrt[4]{243}-\sqrt[4]{96}+2\sqrt[4]{96}$

33) $2\sqrt[4]{2}+2\sqrt[4]{3}+3\sqrt[4]{64}-\sqrt[4]{3}$

34) $2\sqrt[4]{48}-3\sqrt[4]{405}-3\sqrt[4]{48}-\sqrt[4]{162}$

35) $-3\sqrt[5]{6}-\sqrt[5]{64}+2\sqrt[5]{192}-2\sqrt[5]{160}$

36) $2\sqrt[5]{256}-2\sqrt[7]{256}-3\sqrt[7]{2}-\sqrt[7]{640}$

37) $2\sqrt[5]{160}-2\sqrt[5]{192}-\sqrt[5]{160}-\sqrt[5]{-160}$

38) $-2\sqrt[7]{256}-2\sqrt[7]{256}-3\sqrt[7]{2}-\sqrt[7]{640}$

39) $-\sqrt[6]{256}-2\sqrt[6]{4}-3\sqrt[6]{320}-2\sqrt[6]{128}$

40) $-3\sqrt[7]{3}-3\sqrt[7]{768}+2\sqrt[7]{384}+3\sqrt[7]{5}$

8.4

Radicals - Multiply and Divide Radicals

Multiplying radicals is very simple if the index on all the radicals match. The prodcut rule of radicals which we have alread been using can be generalized as follows:

$$\textbf{Product Rule of Radicals: } a\sqrt[m]{b}\cdot c\sqrt[m]{d}=ac\sqrt[m]{bd}$$

Another way of stating this rule is we are allowed to multiply the factors outside the radical and we are allowed to multiply the factors inside the radicals, as long as the index matches. This is shown in the following example.

Example 16.

$-5\sqrt{14}\cdot 4\sqrt{6}$	Multiply outside and inside the radical
$-20\sqrt{84}$	Simplify the radical, divisible by 4
$-20\sqrt{4\cdot 21}$	Take the square root where possible
$-20\cdot 2\sqrt{21}$	Multiply coefficients
$-40\sqrt{21}$	Our Solution

The same process works with higher roots

Example 17.

$2\sqrt[3]{18}\cdot 6\sqrt[3]{15}$	Multiply outside and inside the radical
$12\sqrt[3]{270}$	Simplify the radical, divisible by 27
$12\sqrt[3]{27\cdot 10}$	Take cube root where possible
$12\cdot 3\sqrt[3]{10}$	Multiply coefficients
$36\sqrt[3]{10}$	Our Solution

When multiplying with radicals we can still use the distributive property or FOIL just as we could with variables.

Example 18.

$7\sqrt{6}(3\sqrt{10}-5\sqrt{15})$	Distribute, following rules for multiplying radicals
$21\sqrt{60}-35\sqrt{90}$	Simplify each radical, finding perfect square factors
$21\sqrt{4\cdot 15}-35\sqrt{9\cdot 10}$	Take square root where possible
$21\cdot 2\sqrt{15}-35\cdot 3\sqrt{10}$	Multiply coefficients
$42\sqrt{15}-105\sqrt{10}$	Our Solution

Example 19.

$$\begin{array}{rl}(\sqrt{5}-2\sqrt{3})(4\sqrt{10}+6\sqrt{6}) & \text{FOIL, following rules for multiplying radicals}\\ 4\sqrt{50}+6\sqrt{30}-8\sqrt{30}-12\sqrt{18} & \text{Simplify radicals, find perfect square factors}\\ 4\sqrt{25\cdot 2}+6\sqrt{30}-8\sqrt{30}-12\sqrt{9\cdot 2} & \text{Take square root where possible}\\ 4\cdot 4\sqrt{2}+6\sqrt{30}-8\sqrt{30}-12\cdot 3\sqrt{2} & \text{Multiply coefficients}\\ 16\sqrt{2}+6\sqrt{30}-8\sqrt{30}-36\sqrt{2} & \text{Combine like terms}\\ -20\sqrt{2}-2\sqrt{30} & \text{Our Solution}\end{array}$$

Example 20.

$$\begin{array}{rl}(2\sqrt{5}-3\sqrt{6})(7\sqrt{2}-8\sqrt{7}) & \text{FOIL, following rules for multiplying radicals}\\ 14\sqrt{10}-16\sqrt{35}-21\sqrt{12}-24\sqrt{42} & \text{Simplify radicals, find perfect square factors}\\ 14\sqrt{10}-16\sqrt{35}-21\sqrt{4\cdot 3}-24\sqrt{42} & \text{Take square root where possible}\\ 14\sqrt{10}-16\sqrt{35}-21\cdot 2\sqrt{3}-24\sqrt{42} & \text{Multiply coefficient}\\ 14\sqrt{10}-16\sqrt{35}-42\sqrt{3}-24\sqrt{42} & \text{Our Solution}\end{array}$$

As we are multiplying we always look at our final solution to check if all the radicals are simplified and all like radicals or like terms have been combined.

Division with radicals is very similar to multiplication, if we think about division as reducing fractions, we can reduce the coefficients outside the radicals and reduce the values inside the radicals to get our final solution.

$$\textbf{Quotient Rule of Radicals: } \frac{a\sqrt[m]{b}}{c\sqrt[m]{d}}=\frac{a}{c}\sqrt[m]{\frac{b}{d}}$$

Example 21.

$$\begin{array}{rl}\dfrac{15\sqrt[3]{108}}{20\sqrt[3]{2}} & \text{Reduce } \dfrac{15}{20} \text{ and } \dfrac{\sqrt[3]{108}}{\sqrt{2}} \text{ by dividing by 5 and 2 respectively}\\[2ex] \dfrac{3\sqrt[3]{54}}{4} & \text{Simplify radical, 54 is divisible by 27}\\[2ex] \dfrac{3\sqrt[3]{27\cdot 2}}{4} & \text{Take the cube root of 27}\\[2ex] \dfrac{3\cdot 3\sqrt[3]{2}}{4} & \text{Multiply coefficients}\\[2ex] \dfrac{9\sqrt[3]{2}}{4} & \text{Our Solution}\end{array}$$

There is one catch to dividing with radicals, it is considered bad practice to have a radical in the denominator of our final answer. If there is a radical in the denominator we will rationalize it, or clear out any radicals in the denominator.

We do this by multiplying the numerator and denominator by the same thing. The problems we will consider here will all have a monomial in the denominator. The way we clear a monomial radical in the denominator is to focus on the index. The index tells us how many of each factor we will need to clear the radical. For example, if the index is 4, we will need 4 of each factor to clear the radical. This is shown in the following examples.

Example 22.

$$\frac{\sqrt{6}}{\sqrt{5}} \quad \text{Index is 2, we need two fives in denominator, need 1 more}$$

$$\frac{\sqrt{6}}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) \quad \text{Multiply numerator and denominator by } \sqrt{5}$$

$$\frac{\sqrt{30}}{5} \quad \text{Our Solution}$$

Example 23.

$$\frac{3\sqrt[4]{11}}{\sqrt[4]{2}} \quad \text{Index is 4, we need four twos in denominator, need 3 more}$$

$$\frac{3\sqrt[4]{11}}{\sqrt[4]{2}}\left(\frac{\sqrt[4]{2^3}}{\sqrt[4]{2^3}}\right) \quad \text{Multiply numerator and denominator by } \sqrt[4]{2^3}$$

$$\frac{3\sqrt[4]{88}}{2} \quad \text{Our Solution}$$

Example 24.

$$\frac{4\sqrt[3]{2}}{7\sqrt[3]{25}} \quad \text{The 25 can be written as } 5^2\text{. This will help us keep the numbers small}$$

$$\frac{4\sqrt[3]{2}}{7\sqrt[3]{5^2}} \quad \text{Index is 3, we need three fives in denominator, need 1 more}$$

$$\frac{4\sqrt[3]{2}}{7\sqrt[3]{5^2}}\left(\frac{\sqrt[3]{5}}{\sqrt[3]{5}}\right) \quad \text{Multiply numerator and denominator by } \sqrt[3]{5}$$

$$\frac{4\sqrt[3]{10}}{7\cdot 5} \quad \text{Multiply out denominator}$$

$$\frac{4\sqrt[3]{10}}{35} \quad \text{Our Solution}$$

The previous example could have been solved by multiplying numerator and denominator by $\sqrt[3]{25^2}$. However, this would have made the numbers very large and we would have needed to reduce our soultion at the end. This is why re-writing the radical as $\sqrt[3]{5^2}$ and multiplying by just $\sqrt[3]{5}$ was the better way to simplify.

We will also always want to reduce our fractions (inside and out of the radical) before we rationalize.

Example 25.

$$\frac{6\sqrt{14}}{12\sqrt{22}} \quad \text{Reduce coefficients and inside radical}$$

$$\frac{\sqrt{7}}{2\sqrt{11}} \quad \text{Index is 2, need two elevens, need 1 more}$$

$$\frac{\sqrt{7}}{2\sqrt{11}}\left(\frac{\sqrt{11}}{\sqrt{11}}\right) \quad \text{Multiply numerator and denominator by } \sqrt{11}$$

$$\frac{\sqrt{77}}{2\cdot 11} \quad \text{Multiply denominator}$$

$$\frac{\sqrt{77}}{22} \quad \text{Our Solution}$$

The same process can be used to rationalize fractions with variables.

Example 26.

$$\frac{18\sqrt[4]{6x^3y^4z}}{8\sqrt[4]{10xy^6z^3}} \quad \text{Reduce coefficients and inside radical}$$

$$\frac{9\sqrt[4]{3x^2}}{4\sqrt[4]{5y^2z^3}} \quad \begin{array}{l}\text{Index is 4. We need four of everything to rationalize,}\\ \text{three more fives, two more } y's \text{ and one more } z.\end{array}$$

$$\frac{9\sqrt[4]{3x^2}}{4\sqrt[4]{5y^2z^3}}\left(\frac{\sqrt[4]{5^3y^2z}}{\sqrt[4]{5^3y^2z}}\right) \quad \text{Multiply numerator and denominator by } \sqrt[4]{5^3y^2z}$$

$$\frac{9\sqrt[4]{375x^2y^2z}}{4\cdot 5yz} \quad \text{Multiply denominator}$$

$$\frac{9\sqrt[4]{375x^2y^2z}}{20yz} \quad \text{Our Solution}$$

Practice - Multiply and Divide Radicals

State if the given functions are inverses.

1) $3\sqrt{5}\cdot -4\sqrt{16}$

2) $-5\sqrt{10}\cdot\sqrt{15}$

3) $\sqrt{12m}\cdot\sqrt{15m}$

4) $\sqrt{5r^3}\cdot -5\sqrt{10r^2}$

5) $\sqrt[3]{4x^3}\cdot\sqrt[3]{2x^4}$

6) $3\sqrt[3]{4a^4}\cdot\sqrt[3]{10a^3}$

7) $\sqrt{6}(\sqrt{2}+2)$

8) $\sqrt{10}(\sqrt{5}+\sqrt{2})$

9) $-5\sqrt{15}(3\sqrt{3}+2)$

10) $5\sqrt{15}(3\sqrt{3}+2)$

11) $5\sqrt{10}(5n+\sqrt{2})$

12) $\sqrt{15}(\sqrt{5}-3\sqrt{3v})$

13) $(2+2\sqrt{2})(-3+\sqrt{2})$

14) $(-2+\sqrt{3})(-5+2\sqrt{3})$

15) $(\sqrt{5}-5)(2\sqrt{5}-1)$

16) $(2\sqrt{3}+\sqrt{5})(5\sqrt{3}+2\sqrt{4})$

17) $(\sqrt{2a}+2\sqrt{3a})(3\sqrt{2a}+\sqrt{5a})$

18) $(-2\sqrt{2p}+5\sqrt{5})(\sqrt{5p}+\sqrt{5p})$

19) $(-5-4\sqrt{3})(-3-4\sqrt{3})$

20) $(5\sqrt{2}-1)(-\sqrt{2m}+5)$

21) $\frac{\sqrt{12}}{5\sqrt{100}}$

22) $\frac{\sqrt{15}}{2\sqrt{4}}$

23) $\frac{\sqrt{5}}{4\sqrt{125}}$

24) $\frac{\sqrt{12}}{\sqrt{3}}$

25) $\frac{\sqrt{10}}{\sqrt{6}}$

26) $\frac{\sqrt{2}}{3\sqrt{5}}$

27) $\frac{2\sqrt{4}}{3\sqrt{3}}$

28) $\frac{4\sqrt{3}}{\sqrt{15}}$

29) $\frac{5x^2}{4\sqrt{3x^3y^3}}$

30) $\frac{4}{5\sqrt{3xy^4}}$

31) $\frac{\sqrt{2p^2}}{\sqrt{3p}}$

32) $\frac{\sqrt{8n^2}}{\sqrt{10n}}$

33) $\frac{3\sqrt[3]{10}}{5\sqrt[3]{27}}$

34) $\frac{\sqrt[3]{15}}{\sqrt[3]{64}}$

35) $\frac{\sqrt[3]{5}}{4\sqrt[3]{4}}$

36) $\frac{\sqrt[4]{2}}{2\sqrt[4]{64}}$

37) $\frac{5\sqrt[4]{5r^4}}{\sqrt[4]{8r^2}}$

38) $\frac{4}{\sqrt[4]{65m^4n^2}}$

8.5

Radicals - Rationalize Denominators

It is considered bad practice to have a radical in the denominator of a fraction. When this happens we multiply the numerator and denominator by the same thing in order to clear the radical. In the lesson on dividing radicals we talked about how this was done with monomials. Here we will look at how this is done with binomials.

If the binomial is in the numerator the process to rationalize the denominator is essentially the same as with monomials. The only difference is we will have to distribute in the numerator.

Example 27.

$$\frac{\sqrt{3}-9}{2\sqrt{6}} \quad \text{Want to clear } \sqrt{6} \text{ in denominator, multiply by } \sqrt{6}$$

$$\frac{(\sqrt{3}-9)}{2\sqrt{6}}\left(\frac{\sqrt{6}}{\sqrt{6}}\right) \quad \text{We will distribute the } \sqrt{6} \text{ through the numerator}$$

$$\frac{\sqrt{18}-9\sqrt{6}}{2\cdot 6}$$ Simplify radicals in numerator, multiply out denominator

$$\frac{\sqrt{9\cdot 2}-9\sqrt{6}}{12}$$ Take square root where possible

$$\frac{3\sqrt{2}-9\sqrt{6}}{12}$$ Reduce by dividing each term by 3

$$\frac{\sqrt{2}-3\sqrt{6}}{4}$$ Our Solution

It is important to remember that when reducing the fraction we cannot reduce with just the 3 and 12 or just the 9 and 12. When we have addition or subtraction in the numerator or denominator we must divide all terms by the same number.

The problem can often be made easier if we first simplify any radicals in the problem.

$$\frac{2\sqrt{20x^5}-\sqrt{12x^2}}{\sqrt{18x}}$$ Simplify radicals by finding perfect squares

$$\frac{2\sqrt{4\cdot 5x^3}-\sqrt{4\cdot 3x^2}}{\sqrt{9\cdot 2x}}$$ Simplify roots, divide exponents by 2.

$$\frac{2\cdot 2x^2\sqrt{5x}-2x\sqrt{3}}{3\sqrt{2x}}$$ Multiply coefficients

$$\frac{4x^2\sqrt{5x}-2x\sqrt{3}}{3\sqrt{2x}}$$ Multiplying numerator and denominator by $\sqrt{2x}$

$$\frac{(4x^2\sqrt{5x}-2x\sqrt{3})}{3\sqrt{2x}}\left(\frac{\sqrt{2x}}{\sqrt{2x}}\right)$$ Distribute through numerator

$$\frac{4x^2\sqrt{10x^2}-2x\sqrt{6x}}{3\cdot 2x}$$ Simplify roots in numerator,
multiply coefficients in denominator

$$\frac{4x^3\sqrt{10}-2x\sqrt{6x}}{6x}$$ Reduce, dividing each term by $2x$

$$\frac{2x^2\sqrt{10}-\sqrt{6x}}{3x} \quad \text{Our Solution}$$

As we are rationalizing it will always be important to constantly check our problem to see if it can be simplified more. We ask ourselves, can the fraction be reduced? can the radicals be simplified? These steps may happen several times on our way to the solution.

If the binomial occurs in the denominator we will have to use a different strategy to clear the radical. Consider $\frac{2}{\sqrt{3}-5}$, if we were to multiply the denominator by $\sqrt{3}$ we would have to distribute it and we would end up with $3-5\sqrt{3}$. We haven't cleared the radical, only moved it to another part of the denominator. So our current method will not work. Instead we will use what is called a conjugage. A **conjugate** is made up of the same terms, with the opposite sign in the middle. So for our example with $\sqrt{3}-5$ in the denominator, the conjugate would be $\sqrt{3}+5$. The advantage of a conjugate is when we multiply them together we have $(\sqrt{3}-5)(\sqrt{3}+5)$, which is a sum and a difference. We know when we multiply these we get a difference of squares. Squaring $\sqrt{3}$ and 5, with subtraction in the middle gives the product $3-25=-22$. Our answer when multiplying conjugates will no longer have a square root. This is exactly what we want.

Example 28.

$$\frac{2}{\sqrt{3}-5} \quad \text{Multiply numerator and denominator by conjugate}$$

$$\frac{2}{\sqrt{3}-5}\left(\frac{\sqrt{3}+5}{\sqrt{3}+5}\right) \quad \text{Distribute numerator, difference of squares in denominator}$$

$$\frac{2\sqrt{3}+10}{3-25} \quad \text{Simplify denoinator}$$

$$\frac{2\sqrt{3}+10}{-22} \quad \text{Reduce by dividing all terms by} -2$$

$$\frac{-\sqrt{3}-5}{11} \quad \text{Our Solution}$$

In the previous example, we could have reduced by dividng by 2, giving the solution $\frac{\sqrt{3}+5}{-11}$, both answers are correct.

Example 29.

$$\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}} \quad \text{Multilpy by conjugate, } \sqrt{5}-\sqrt{3}$$

$$\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right) \quad \text{Distribute numerator, denominator is difference of squares}$$

$$\frac{\sqrt{75}-\sqrt{45}}{5-3} \quad \text{Simplify radicals in numerator, subtract in denominator}$$

$$\frac{\sqrt{25\cdot 3}-\sqrt{9\cdot 5}}{2} \quad \text{Take square roots where possible}$$

$$\frac{5\sqrt{3}-3\sqrt{5}}{2} \quad \text{Our Solution}$$

Example 30.

$$\frac{2\sqrt{3x}}{4-\sqrt{5x^3}} \quad \text{Multiply by conjugate, } 4+\sqrt{5x^3}$$

$$\frac{2\sqrt{3x}}{4-\sqrt{5x^3}}\left(\frac{4+\sqrt{5x^3}}{4+\sqrt{5x^3}}\right) \quad \text{Distribute numerator, denominator is difference of squares}$$

$$\frac{8\sqrt{3x}+2\sqrt{15x^4}}{16-5x^3} \quad \text{Simplify radicals where possible}$$

$$\frac{8\sqrt{3x}+2x^2\sqrt{15}}{16-5x^3} \quad \text{Our Solution}$$

The same process can be used when there is a binomial in the numerator and denominator. We just need to remember to FOIL out the numerator.

Example 31.

$$\frac{3-\sqrt{5}}{2-\sqrt{3}} \quad \text{Multiply by conjugate, } 2+\sqrt{3}$$

$$\frac{3-\sqrt{5}}{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right) \quad \text{FOIL in numerator, denominator is difference of squares}$$

$$\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{4-3} \quad \text{Simplify denominator}$$

$$\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{1} \quad \text{Divide each term by 1}$$

$6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}$ Our Solution

Example 32.

$$\frac{2\sqrt{5}-3\sqrt{7}}{5\sqrt{6}+4\sqrt{2}} \quad \text{Multiply by the conjugate, } 5\sqrt{6}-4\sqrt{2}$$

$$\frac{2\sqrt{5}-3\sqrt{7}}{5\sqrt{6}+4\sqrt{2}}\left(\frac{5\sqrt{6}-4\sqrt{2}}{5\sqrt{6}-4\sqrt{2}}\right) \quad \text{FOIL numerator, denominator is difference of squares}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}+12\sqrt{14}}{25\cdot 6-16\cdot 2} \quad \text{Multiply in denominator}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}+12\sqrt{14}}{150-32} \quad \text{Subtract in denominator}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}+12\sqrt{14}}{118} \quad \text{Our Solution}$$

The same process is used when we have variables

Example 33.

$$\frac{3x\sqrt{2x}+\sqrt{4x^3}}{5x-\sqrt{3x}} \quad \text{Multiply by the conjugate, } 5x+\sqrt{3x}$$

$$\frac{3x\sqrt{2x}+\sqrt{4x^3}}{5x-\sqrt{3x}}\left(\frac{5x+\sqrt{3x}}{5x+\sqrt{3x}}\right) \quad \text{FOIL in numerator, denominator is difference of squares}$$

$$\frac{15x^2\sqrt{2x}+3x\sqrt{6x^2}+5x\sqrt{4x^3}+\sqrt{12x^4}}{25x^2-3x} \quad \text{Simplify radicals}$$

$$\frac{15x^2\sqrt{2x}+3x^2\sqrt{6}+10x^2\sqrt{x}+2x^2\sqrt{3}}{25x^2-3x} \quad \text{Divide each term by } x$$

$$\frac{15x\sqrt{2x}+3x\sqrt{6}+10x\sqrt{x}+2x\sqrt{3}}{25x-3} \quad \text{Our Solution}$$

Practice - Rationalize Denominators

Simplify.

1) $\frac{4+2\sqrt{3}}{\sqrt{9}}$

3) $\frac{4+2\sqrt{3}}{5\sqrt{4}}$

5) $\frac{2-5\sqrt{5}}{4\sqrt{13}}$

7) $\frac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}$

9) $\frac{2p+3\sqrt{5p^4}}{5\sqrt{20p^2}}$

11) $\frac{\sqrt{3m^2}-4\sqrt{2m^4}}{5\sqrt{12m^4}}$

13) $\frac{5}{3\sqrt{5}+\sqrt{2}}$

15) $\frac{2}{5+\sqrt{2}}$

17) $\frac{3}{4-3\sqrt{3}}$

19) $\frac{4}{3+\sqrt{5}}$

21) $-\frac{4}{4-4\sqrt{2}}$

23) $\frac{5}{\sqrt{n^4}-5}$

25) $\frac{4p}{3-5\sqrt{p^4}}$

27) $\frac{4}{5+\sqrt{5x^2}}$

29) $\frac{5}{2+\sqrt{5r^3}}$

31) $\frac{5}{-5v-3\sqrt{v}}$

33) $\frac{4\sqrt{2}+3}{3\sqrt{2}+\sqrt{3}}$

35) $\frac{2-\sqrt{5}}{-3+\sqrt{5}}$

37) $\frac{5\sqrt{2}+\sqrt{3}}{5+5\sqrt{2}}$

39) $\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$

41) $\frac{\sqrt{3}-\sqrt{2}}{4+\sqrt{5}}$

43) $\frac{4+2\sqrt{2x^2}}{5+2\sqrt{5x^3}}$

45) $\frac{2\sqrt{3m^2}-\sqrt{2m^4}}{5-\sqrt{3m^2}}$

47) $\frac{2b-5\sqrt{2b}}{-1+\sqrt{2b^4}}$

49) $\frac{2-\sqrt{2x}}{4x-5\sqrt{3x^3}}$

51) $\frac{-4p-\sqrt{p}}{-p-\sqrt{p^3}}$

2) $\frac{-4+\sqrt{3}}{4\sqrt{9}}$

4) $\frac{2\sqrt{3}-2}{2\sqrt{16}}$

6) $\frac{\sqrt{5}+4}{4\sqrt{17}}$

8) $\frac{\sqrt{5}-\sqrt{2}}{3\sqrt{6}}$

10) $\frac{5x-\sqrt{3x^4}}{2\sqrt{19x^2}}$

12) $\frac{3\sqrt{2r^4}-\sqrt{3r^4}}{\sqrt{14r^3}}$

14) $\frac{5}{\sqrt{3}+4\sqrt{5}}$

16) $\frac{5}{2\sqrt{3}-\sqrt{2}}$

18) $\frac{4}{\sqrt{2}-2}$

20) $\frac{2}{2\sqrt{5}+2\sqrt{3}}$

22) $\frac{4}{4\sqrt{3}-\sqrt{5}}$

24) $\frac{\sqrt{3n^3}}{-5-\sqrt{2n^4}}$

26) $\frac{5x^2}{5-3\sqrt{5x}}$

28) $\frac{4b^3}{-4b^4-2\sqrt{b^4}}$

30) $\frac{5}{\sqrt{3a^4}+4a}$

32) $\frac{4}{2\sqrt{2n^2}+\sqrt{n^3}}$

34) $\frac{-5-4\sqrt{5}}{5-5\sqrt{3}}$

36) $\frac{-1+\sqrt{5}}{2\sqrt{5}+5\sqrt{2}}$

38) $\frac{4\sqrt{2}+2\sqrt{3}}{\sqrt{5}-4}$

40) $\frac{\sqrt{3}-\sqrt{5}}{4-\sqrt{2}}$

42) $\frac{2\sqrt{2}-\sqrt{3}}{\sqrt{5}-3\sqrt{3}}$

44) $\frac{3\sqrt{p^4}+\sqrt{5p^3}}{4-\sqrt{2p^4}}$

46) $\frac{4v-2\sqrt{2v^2}}{-4-5\sqrt{v}}$

48) $\frac{-4n-\sqrt{3n^2}}{-3-\sqrt{n^4}}$

50) $\frac{3a+2\sqrt{2a^2}}{3-5\sqrt{3a^2}}$

52) $\frac{4-\sqrt{5x}}{-2x-\sqrt{3x^2}}$

8.6

Radicals - Rational Exponents

When we simplify radicals with exponents, we divide the exponent by the index. Another way to write division with with a fraction bar. This idea is how we will define rational exponents.

Definition of Rational Exponents: $\boldsymbol{a^{\frac{n}{m}} = (\sqrt[m]{a})^n}$

The denominator of a rational exponent becomes the index on our radical, likewise the index on the radical becomes the denominator of the exponent. We can use this property to change any radical expression into an exponential expression.

Example 34.

$(\sqrt[5]{x})^3 = x^{\frac{5}{5}}$	$(\sqrt[6]{3x})^5 = (3x)^{\frac{5}{6}}$
$\dfrac{1}{(\sqrt[7]{a})^3} = a^{-\frac{3}{7}}$	$\dfrac{1}{(\sqrt[3]{xy})^2} = (xy)^{-\frac{2}{3}}$

Index is denominator
Negative exponents from reciprocals

We can also change any rational expoennt into a radical expression by using the denominator as the index.

Example 35.

$a^{\frac{5}{3}} = (\sqrt[3]{a})^5$	$(2mn)^{\frac{2}{7}} = (\sqrt[7]{2mn})^2$
$x^{-\frac{4}{5}} = \dfrac{1}{(\sqrt[5]{x})^4}$	$(xy)^{-\frac{2}{9}} = \dfrac{1}{(\sqrt[9]{xy})^2}$

Index is denominator
Negative exponent means reciprocals

The ability to change between exponential expressions and radical expressions allows us to evaluate problems we had no means of evaluating before by changing to a radcal.

Example 36.

$27^{-\frac{4}{3}}$ Change to radical, denominator is index, negative means reciprocal

$\dfrac{1}{(\sqrt[3]{27})^4}$ Evaluate radical

$\dfrac{1}{(3)^4}$ Evaluate exponent

$$\frac{1}{81} \quad \text{Our solution}$$

The largest advantage of being able to change a radical expression into an exponential expression is we are now allowed to use all our exponent properties to simplify. The following table reviews all of our exponent properties.

Properties of Exponents

$$a^m a^n = a^{m+n} \qquad (ab)^m = a^m b^m \qquad a^{-m} = \frac{1}{a^m}$$

$$\frac{a^m}{a^n} = a^{m-n} \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \qquad \frac{1}{a^{-m}} = a^m$$

$$(a^m)^n = a^{mn} \qquad a^0 = 1 \qquad \left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}$$

When adding and subtracting with fractions we need to be sure to have a common denominator. When multiplying we only need to multiply the numerators together and denominators together. The following examples show several different problems, using different properties to simplify the rational exponents.

Example 37.

$$a^{\frac{2}{3}} b^{\frac{1}{2}} a^{\frac{1}{6}} b^{\frac{1}{5}} \quad \text{Need common denominator on } a's\,(6) \text{ and } b's\,(10)$$

$$a^{\frac{4}{6}} b^{\frac{5}{10}} a^{\frac{1}{6}} b^{\frac{2}{10}} \quad \text{Add exponents on } a's \text{ and } b's$$

$$a^{\frac{5}{6}} b^{\frac{7}{10}} \quad \text{Our Solution}$$

Example 38.

$$\left(x^{\frac{1}{3}} y^{\frac{2}{5}}\right)^{\frac{3}{4}} \quad \text{Multiply } \frac{3}{4} \text{ by each exponent}$$

$$x^{\frac{1}{4}} y^{\frac{3}{10}} \quad \text{Our Solution}$$

Example 39.

$$\frac{x^2 y^{\frac{2}{3}} \cdot 2x^{\frac{1}{2}} y^{\frac{5}{6}}}{x^{\frac{7}{2}} y^0} \quad \text{In numerator, need common denominator to add exponents}$$

$$\frac{x^{\frac{4}{2}}y^{\frac{4}{6}}\cdot 2x^{\frac{1}{2}}y^{\frac{5}{6}}}{x^{\frac{7}{2}}y^0}$$ Add exponents in numerator, in denominator, $y^0=1$

$$\frac{2x^{\frac{5}{2}}y^{\frac{9}{6}}}{x^{\frac{7}{2}}}$$ Subtract exponents on x, reduce exponent on y

$$2x^{-1}y^{\frac{3}{2}}$$ Negative exponent moves down to denominator

$$\frac{2y^{\frac{3}{2}}}{x}$$ Our Solution

Example 40.

$$\left(\frac{25x^{\frac{1}{3}}y^{\frac{2}{5}}}{9x^{\frac{4}{5}}y^{-\frac{3}{2}}}\right)^{-\frac{1}{2}}$$ Using order of operations, simplify inside parenthesis first
Need common denominators before we can subtract exponents

$$\left(\frac{25x^{\frac{5}{15}}y^{\frac{4}{10}}}{9x^{\frac{12}{15}}y^{-\frac{15}{10}}}\right)^{-\frac{1}{2}}$$ Subtract exponents, be careful of the negative:
$\frac{4}{10}-\left(-\frac{15}{10}\right)=\frac{4}{10}+\frac{15}{10}=\frac{19}{10}$

$$\left(\frac{25x^{-\frac{7}{15}}y^{\frac{19}{10}}}{9}\right)^{-\frac{1}{2}}$$ The negative exponent will flip the fraction

$$\left(\frac{9}{25x^{-\frac{7}{15}}y^{\frac{19}{10}}}\right)^{\frac{1}{2}}$$ The exponent $\frac{1}{2}$ goes on each factor

$$\frac{9^{\frac{1}{2}}}{25^{\frac{1}{2}}x^{-\frac{7}{30}}y^{\frac{19}{20}}}$$ Evaluate $9^{\frac{1}{2}}$ and $25^{\frac{1}{2}}$ and move negative exponent

$$\frac{3x^{\frac{7}{30}}}{5y^{\frac{19}{20}}}$$ Our Solution

It is important to remember that as we simplify with rational exponents we are using the exact same properties we used when simplifying integer exponents. The only difference is we need to follow our rules for fractions as well. It may be worth reviewing your notes on exponent properties to be sure your comfortable with using the properties.

Practice - Rational Exponents

Write each expression in radical form.

1) $m^{\frac{3}{5}}$

2) $(10r)^{-\frac{3}{4}}$

3) $(7x)^{\frac{3}{2}}$

4) $(6b)^{-\frac{4}{3}}$

Write each expression in exponential form.

5) $\frac{1}{(\sqrt{6x})^3}$

6) $\sqrt{v}$

7) $\frac{1}{(\sqrt[4]{n})^7}$

8) $\sqrt{5a}$

Simplify. Your answer should contain only positive exponents.

13) $yx^{\frac{1}{3}} \cdot xy^{\frac{3}{2}}$

14) $4v^{\frac{2}{3}} \cdot v^{-1}$

15) $(a^{\frac{1}{2}}b^{\frac{1}{2}})^{-1}$

16) $(x^{\frac{5}{3}}y^{-2})^0$

17) $\frac{a^2b^0}{3a^4}$

18) $\frac{2x^{\frac{1}{2}}y^{\frac{1}{3}}}{2x^{\frac{4}{3}}y^{-\frac{7}{4}}}$

19) $uv \cdot u \cdot (v^{\frac{3}{2}})^3$

20) $(x \cdot xy^2)^0$

21) $(x^0y^{\frac{1}{3}})^{\frac{3}{2}}x^0$

22) $u^{-\frac{5}{4}}v^2 \cdot (u^{\frac{3}{2}})^{-\frac{3}{2}}$

23) $\frac{a^{\frac{3}{4}}b^{-1} \cdot b^{\frac{7}{4}}}{3b^{-1}}$

24) $\frac{2x^{-2}y^{\frac{5}{3}}}{x^{-\frac{5}{4}}y^{-\frac{5}{3}} \cdot xy^{\frac{1}{2}}}$

25) $\frac{3y^{-\frac{5}{4}}}{y^{-1} \cdot 2y^{-\frac{1}{3}}}$

26) $\frac{ab^{\frac{1}{3}} \cdot 2b^{-\frac{5}{4}}}{4a^{-\frac{1}{2}}b^{-\frac{2}{3}}}$

27) $\left(\frac{m^{\frac{3}{2}}n^{-2}}{(mn^{\frac{4}{3}})^{-1}}\right)^{\frac{7}{4}}$

28) $\frac{(y^{-\frac{1}{2}})^{\frac{3}{2}}}{x^{\frac{3}{2}}y^{\frac{1}{2}}}$

29) $\frac{(m^2n^{\frac{1}{2}})^0}{n^{\frac{3}{4}}}$

30) $\frac{y^0}{(x^{\frac{3}{4}}y^{-1})^{\frac{1}{3}}}$

31) $\frac{(x^{-\frac{4}{3}}y^{-\frac{1}{3}} \cdot y)^{-1}}{x^{\frac{1}{3}}y^{-2}}$

32) $\frac{(x^{\frac{1}{2}}y^0)^{-\frac{4}{3}}}{y^4 \cdot x^{-2}y^{-\frac{2}{3}}}$

33) $\frac{(uv^2)^{\frac{1}{2}}}{v^{-\frac{1}{4}}v^2}$

34) $\left(\frac{y^{\frac{1}{3}}y^{-2}}{(x^{\frac{5}{3}}y^3)^{-\frac{3}{2}}}\right)^{\frac{3}{2}}$

8.7

Radicals - Mixed Index

Knowing that a radical has the same properties as exponents (written as a ratio) allows us to manipulate radicals in new ways. One thing we are allowed to do is reduce, not just the radicand, but the index as well. This is shown in the following example.

Example 41.

$\sqrt[8]{x^6y^2}$	Rewrite as raitonal exponent
$(x^6y^2)^{\frac{1}{5}}$	Multiply exponents
$x^{\frac{6}{8}}y^{\frac{2}{8}}$	Reduce each fraction
$x^{\frac{3}{4}}y^{\frac{1}{4}}$	All exponents have denominator of 4, this is our new index
$\sqrt[4]{x^3y}$	Our Solution

What we have done is reduced our index by dividing the index and all the exponents by the same number (2 in the previous example). If we notice a common factor in the index and all the exponnets on every factor we can reduce by dividing by that common factor. This is shown in the next example

Example 42.

$\sqrt[24]{a^6b^9c^{15}}$	Index and all exponents are divisible by 3
$\sqrt[8]{a^2b^3c^5}$	Our Solution

We can use the same process when there are coefficients in the problem. We will first write the coefficient as an exponential expression so we can divide the exponet by the common factor as well.

Example 43.

$\sqrt[9]{8m^6n^3}$	Write 8 as 2^3
$\sqrt[9]{2^3m^6n^3}$	Index and all exponents are divisible by 3
$\sqrt[3]{2m^2n}$	Our Solution

We can use a very similar idea to also multiply radicals where the index does not match. First we will consider an example using rational exponents, then identify the pattern we can use.

Example 44.

$$
\begin{array}{rl}
\sqrt[3]{ab^2}\,\sqrt[4]{a^2b} & \text{Rewrite as rational exponents} \\
(ab^2)^{\frac{1}{3}}(a^2b)^{\frac{1}{4}} & \text{Multiply exponents} \\
a^{\frac{1}{3}}b^{\frac{2}{3}}a^{\frac{2}{4}}b^{\frac{1}{4}} & \text{To have one radical need } a \text{ common denominator, } 12 \\
a^{\frac{4}{12}}b^{\frac{8}{12}}a^{\frac{6}{12}}b^{\frac{3}{12}} & \text{Write under } a \text{ single radical with common index, } 12 \\
\sqrt[12]{a^4b^8a^6b^3} & \text{Add exponents} \\
\sqrt[12]{a^{10}b^{11}} & \text{Our Solution}
\end{array}
$$

To combine the radicals we need a common index (just like the common denominator). We will get a common index by multiplying each index and exponent by an integer that will allow us to build up to that desired index. This process is shown in the next example.

Example 45.

$$
\begin{array}{rl}
\sqrt[4]{a^2b^3}\,\sqrt[6]{a^2b} & \text{Common index is 12.} \\
 & \text{Multiply first index and exponents by 3, second by 2} \\
\sqrt[12]{a^6b^9a^4b^2} & \text{Add exponents} \\
\sqrt[12]{a^{10}b^{11}} & \text{Our Solution}
\end{array}
$$

Often after combining radicals of mixed index we will need to simplify the resulting radical.

Example 46.

$$
\begin{array}{rl}
\sqrt[5]{x^3y^4}\,\sqrt[3]{x^2y} & \text{Common index: 15.} \\
 & \text{Multiply first index and exponents by 3, second by 5} \\
\sqrt[15]{x^9y^{12}x^{10}y^5} & \text{Add exponents} \\
\sqrt[15]{x^{19}y^{17}} & \text{Simplify by dividing exponents by index, remainder is left inside} \\
xy\,\sqrt[15]{x^4y^2} & \text{Our Solution}
\end{array}
$$

Just as with reducing the index, we will rewrite coefficients as exponential expressions. This will also allow us to use exponent properties to simplify.

Example 47.

$\sqrt[3]{4x^2y}\,\sqrt[4]{8xy^3}$	Rewrite 4 as 2^2 and 8 as 2^3
$\sqrt[3]{2^2x^2y}\,\sqrt[4]{2^3xy^3}$	Common index: 12.
	Multiply first index and exponents by 4, second by 3
$\sqrt[12]{2^4x^8y^42^9x^3y^9}$	Add exponents (even on the 2)
$\sqrt[12]{2^{13}x^{11}y^{13}}$	Simplify by dividing exponents by index, remainder is left inside
$2y\,\sqrt[12]{2x^{11}y}$	Our Solution

If there is a binomial in the radical then we need to keep that binomial together through the entire problem.

Example 48.

$\sqrt{3x(y+z)}\,\sqrt[3]{9x(y+z)^2}$	Rewrite 9 as 3^2
$\sqrt{3x(y+z)}\,\sqrt[3]{3^2x(y+z)^2}$	Common index: 6. Multiply first group by 3, second by 2
$\sqrt[6]{3^3x^3(y+z)^33^4x^2(y+z)^4}$	Add exponents, keep $(y+z)$ as binomial
$\sqrt[6]{3^7x^5(y+z)^7}$	Simplify, dividing exponent by index, remainder inside
$3(y+z)\,\sqrt[6]{3x^5(y+z)}$	Our Solution

The same process is used for dividing mixed index as with multilpying mixed index. The only difference is our final answer can't have a radical over the denominator.

Example 49.

$\dfrac{\sqrt[6]{x^4y^3z^2}}{\sqrt[8]{x^7y^2z}}$	Common index is 24. Multiply first group by 4, second by 3
$\sqrt[24]{\dfrac{x^{16}y^{12}z^8}{x^{21}y^6z^3}}$	Subtract exponents
$\sqrt[24]{x^{-5}y^6z^5}$	Negative exponent moves to denominator
$\sqrt[12]{\dfrac{y^6z^5}{x^5}}$	Can$'t$ have denominator in radical, need $12x's$, or 7 more
$\sqrt[12]{\dfrac{y^6z^5}{x^5}}\left(\sqrt[12]{\dfrac{x^7}{x^7}}\right)$	Multiply numerator and denominator by $\sqrt[12]{x^7}$
$\dfrac{\sqrt[12]{x^7y^6z^5}}{x}$	Our Solution

Practice - Radicals of Mixed Index

Reduce the following radicals.

1) $\sqrt[8]{16x^4y^6}$

2) $\sqrt[4]{9x^2y^6}$

3) $\sqrt[12]{64x^4y^6z^8}$

4) $\sqrt[4]{\frac{25x^3}{16x^5}}$

5) $\sqrt[6]{\frac{16x^2}{9y^4}}$

6) $\sqrt[15]{x^9y^{12}z^6}$

7) $\sqrt[12]{x^6y^9}$

8) $\sqrt[10]{64x^8y^4}$

9) $\sqrt[8]{8x^3y^6}$

10) $\sqrt[4]{25y^2}$

11) $\sqrt[9]{8x^3y^6}$

12) $\sqrt[16]{81x^8y^{12}}$

Combine the following radicals.

13) $\sqrt[3]{5}\sqrt{6}$

14) $\sqrt[3]{7}\sqrt[4]{5}$

15) $\sqrt{x}\sqrt[3]{7y}$

16) $\sqrt[3]{y}\ \sqrt[5]{3z}$

17) $\sqrt{x}\sqrt[3]{x-2}$

18) $\sqrt[4]{3x}\ \sqrt{y+4}$

19) $\sqrt[5]{x^2y}\sqrt{xy}$

20) $\sqrt{ab}\ \sqrt[5]{2a^2b^2}$

21) $\sqrt[4]{xy^2}\sqrt[3]{x^2y}$

22) $\sqrt[5]{a^2b^3}\ \sqrt[4]{a^2b}$

23) $\sqrt[4]{a^2bc^2}\sqrt[5]{a^2b^3c}$

24) $\sqrt[6]{x^2yz^3}\ \sqrt[5]{x^2yz^2}$

25) $\sqrt{a}\sqrt[4]{a^3}$

26) $\sqrt[3]{x^2}\ \sqrt[6]{x^5}$

27) $\sqrt[5]{b^2}\sqrt{b^3}$

28) $\sqrt[4]{a^3}\ \sqrt[3]{a^2}$

29) $\sqrt{xy^3}\ \sqrt[3]{x^2y}$

30) $\sqrt[5]{a^3b}\ \sqrt{ab}$

31) $\sqrt[4]{9ab^3}\ \sqrt[4]{9x^3yz^2}$

32) $\sqrt{2x^3y^3}\ \sqrt[3]{4xy^2}$

33) $\sqrt[3]{3xy^2z}\ \sqrt[4]{9x^3yz^2}$

34) $\sqrt{a^4b^3c^4}\ \sqrt[3]{ab^2c}$

35) $\sqrt{27a^5(b+1)}\ \sqrt[3]{81a(b+1)^4}$

36) $\sqrt{8x(y+z)^5}\ \sqrt[3]{4x^2(y+z)^2}$

37) $\frac{\sqrt[3]{a^2}}{\sqrt[4]{a}}$

38) $\frac{\sqrt[3]{x^2}}{\sqrt[5]{x}}$

39) $\frac{\sqrt[4]{x^2y^3}}{\sqrt[3]{xy}}$

40) $\frac{\sqrt[5]{a^4b^2}}{\sqrt[3]{ab^2}}$

41) $\frac{\sqrt{ab^3c}}{\sqrt[5]{a^2b^3c^{-1}}}$

42) $\frac{\sqrt[5]{x^3y^4z^9}}{\sqrt{xy^{-2}z}}$

43) $\frac{\sqrt[4]{(3x-1)^3}}{\sqrt[5]{(3x-1)^3}}$

44) $\frac{\sqrt[3]{(2+5x)^2}}{\sqrt[4]{(2+5x)}}$

45) $\frac{\sqrt[3]{(2x+1)^2}}{\sqrt[5]{(2x+1)^2}}$

46) $\frac{\sqrt[4]{(5-3x)^3}}{\sqrt[3]{(5-3x)^2}}$

8.8

Radicals - Complex Numbers

World View Note: When mathematics was first used the primary purpose was for counting. Thus they did not origionally use negatives, zero, fractions or irrational numbers. However, the ancient Egyptians quickly developed the need for "a part" and so they made up a new type of number, the ratio or fraction. The Ancient Greeks did not believe in irrational numbers (people were killed for believing otherwise). The Mayans of Central America later made up the number zero when they found use for it as a placeholder. Ancient Chinese Mathematicians made up negative numbers when they found use for them.

In mathematics, when the current number system does not provide the tools to solve the problems the culture is working with, we tend to make up new ways for dealing with the problem that can solve the problem. Throughout history this has been the case with the need for a number that is nothing (0), smaller than zero (negatives), between integers (fractions), and between fractions (irrational numbers). This is also the case for the square roots of negative numbers. To work with the square root of negative numbers mathematicians have defined what are called imaginary and complex numbers.

Definition of Imaginary Numbers: $\boldsymbol{i^2 = -1}$ (thus $\boldsymbol{i = \sqrt{-1}}$)

Examples of imaginary numbers include $3i$, $-6i$, $\frac{3}{5}i$ and $3i\sqrt{5}$. A **complex number** is one that contains both a real and imaginary part, such as $2+5i$.

With this definition, the square root of a negative number is no longer undefined. We now are allowed to do basic operations with the square root of negatives. First we will consider exponents on imaginary numbers. We will do this by manipulating our definition of $i^2 = -1$. If we multiply both sides of the definition by i, the equation becomes $i^3 = -i$. Then if we multiply both sides of the equation again by i, the equation becomes $i^4 = -i^2 = -(-1) = 1$, or simply $i^4 = 1$. Multiplying again by i gives $i^5 = i$. One more time gives $i^6 = i^2 = -1$. And if this pattern continues we see a cycle forming, the exponents on i change we cycle through simplified answers of i, -1, $-i$, 1. As there are 4 different possible answers in this cycle, if we divide the exponent by 4 and consider the remainder, we can simplify any exponent on i by learning just the following four values:

Cyclic Property of Powers of $\boldsymbol{i}$

$$
\begin{aligned}
&i^0 = 1\\
&i = i\\
&i^2 = -1\\
&i^3 = -i
\end{aligned}
$$

Example 50.

$$\begin{array}{rl} i^{35} & \text{Divide exponent by 4} \\ 8\,R3 & \text{Use remainder as exponent on } i \\ i^3 & \text{Simplify} \\ -i & \text{Our Solution} \end{array}$$

Example 51.

$$\begin{array}{rl} i^{124} & \text{Divide exponent by 4} \\ 31\,R0 & \text{Use remainder as exponent on } i \\ i^0 & \text{Simplify} \\ 1 & \text{Our Solution} \end{array}$$

When performing operations (add, subtract, multilpy, divide) we can handel i just like we handel any other variable. This means when adding and subtracting complex numbers we simply add or combine like terms.

Example 52.

$$\begin{array}{rl} (2+5i)+(4-7i) & \text{Combine like terms } 2+4 \text{ and } 5i-7i \\ 6-2i & \text{Our Solution} \end{array}$$

It is important to notice what operation we are doing. Students often see the parenthesis and think that means FOIL. We only use FOIL to multiply. This problem is an addition problem so we simply add the terms, or combine like terms.

For subtraction problems the idea is the same, we need to remember to first distribute the negative onto all the terms in the parentheses.

Example 53.

$$\begin{array}{rl} (4-8i)-(3-5i) & \text{Distribute the negative} \\ 4-8i-3+5i & \text{Combine like terms } 4-2 \text{ and } -8i+5i \\ 2-3i & \text{Our Solution} \end{array}$$

Addition and subtraction can be combined into one problem.

Example 54.

$$\begin{array}{rl} (5i)-(3+8i)+(-4+7i) & \text{Distribute the negative} \\ 5i-3-8i-4+7i & \text{Combine like terms } 5i-8i+7i \text{ and } -3-4 \\ -7+4i & \text{Our Solution} \end{array}$$

Multiplying with complex numbers is the same as multiplying with variables with one exception, we will want to simplify our final answer so there are no exponents on i.

Example 55.

$$\begin{array}{rl} (3i)(7i) & \text{Multilpy coefficients and } i's \\ 21i^2 & \text{Simplify } i^2 = -1 \\ 21(-1) & \text{Multiply} \\ -21 & \text{Our Solution} \end{array}$$

Example 56.

$$\begin{array}{rl} 5i(3i-7) & \text{Distribute} \\ 15i^2 - 35i & \text{Simplify } i^2 = -1 \\ 15(-1) - 35i & \text{Multiply} \\ -15 - 35i & \text{Our Solution} \end{array}$$

Example 57.

$$\begin{array}{rl} (2-4i)(3+5i) & \text{FOIL} \\ 6 + 10i - 12i - 20i^2 & \text{Simplify } i^2 = -1 \\ 6 + 10i - 12i - 20(-1) & \text{Multiply} \\ 6 + 10i - 12i + 20 & \text{Combine like trems } 6 + 20 \text{ and } 10i - 12i \\ 26 - 2i & \text{Our Solution} \end{array}$$

Example 58.

$$\begin{array}{rl} (3i)(6i)(2-3i) & \text{Multiply first two monomials} \\ 18i^2(2-3i) & \text{Distribute} \\ 36i^2 - 48i^3 & \text{Simplify } i^2 = -1 \text{ and } i^3 = -i \\ 36(-1) - 48(-i) & \text{Multiply} \\ -36 + 48i & \text{Our Solution} \end{array}$$

Remember when squaring a binomial we either have to FOIL or use our shortcut to square the first, twice the product and square the last. The next example uses the shortcut

Example 59.

$$\begin{array}{rl} (4-5i)^2 & \text{Use perfect square shortcut} \\ 4^2 = 16 & \text{Square the first} \\ 2(4)(-5i) = -40i & \text{Twice the product} \\ (5i)^2 = 25i^2 = 25(-1) = -25 & \text{Square the last, simplify } i^2 = -1 \\ 16 - 40i - 25 & \text{Combine like terms} \\ -9 - 40i & \text{Our Solution} \end{array}$$

Dividing with complex numbers also has one thing we need to be careful of. If i is $\sqrt{-1}$, and it is in the denominator of a fraction, then we have a radical in the denominator! This means we will want to rationalize our denominator so there are no i's. This is done the same way we rationalized denoinators with square roots.

Example 60.

$$\frac{7+3i}{-5i} \quad \text{Just } a \text{ monomial in denominator, multiply by } i$$

$$\frac{7+3i}{-5i}\left(\frac{i}{i}\right) \quad \text{Distribute } i \text{ in numerator}$$

$$\frac{7i+3i^2}{-5i^2} \quad \text{Simplify } i^2=-1$$

$$\frac{7i+3(-1)}{-5(-1)} \quad \text{Multiply}$$

$$\frac{7i-3}{5} \quad \text{Our Solution}$$

The solution for these problems can be written several different ways, for example $\frac{-3+7i}{5}$ or $\frac{-3}{5}+\frac{7}{5}i$, The author has elected to use the solution as written, but it is important to express your answer in the form your instructor prefers.

Example 61.

$$\frac{2-6i}{4+8i} \quad \text{Binomial in denominator, multiply by conjugate, } 4-8i$$

$$\frac{2-6i}{4+8i}\left(\frac{4-8i}{4-8i}\right) \quad \text{FOIL in numerator, denominator is } a \text{ difference of squares}$$

$$\frac{8-16i-24i+48i^2}{16-64i^2} \quad \text{Simplify } i^2=-1$$

$$\frac{8-16i-24i+48(-1)}{16-64(-1)} \quad \text{Multiply}$$

$$\frac{8-16i-24i-48}{16+64} \quad \text{Combine like terms } 8-48 \text{ and } -16i-24i \text{ and } 16+64$$

$$\frac{-40-40i}{80} \quad \text{Reduce, divide each term by } 40$$

$$\frac{-1-i}{2} \quad \text{Our Solution}$$

Practice - Complex Numbers

Simplify.

1) $3-(-8+4i)$

2) $(3i)-(7i)$

3) $(7i)-(3-2i)$

4) $5+(-6-6i)$

5) $(-6i)-(3+7i)$

6) $(-8i)-(7i)-(5-3i)$

7) $(3-3i)+(-7-8i)$

8) $(-4-i)+(1-5i)$

9) $(i)-(2+3i)-6$

10) $(5-4i)+(8-4i)$

11) $(6i)(-8i)$

12) $(3i)(-8i)$

13) $(-5i)(8i)$

14) $(8i)(-4i)$

15) $(-7i)^2$

16) $(-i)(7i)(4-3i)$

17) $(6+5i)^2$

18) $(8i)(-2i)(-2-8i)$

19) $(-7-4i)(-8+6i)$

20) $(3i)(-3i)(4-4i)$

21) $(-4+5i)(2-7i)$

22) $-8(4-8i)-2(-2-6i)$

23) $(-8-6i)(-4+2i)$

24) $(-6i)(3-2i)-(7i)(4i)$

25) $(1+5i)(2+i)$

26) $(-2+i)(3-5i)$

27) $\frac{-9+5i}{i}$

28) $\frac{-3+2i}{-3i}$

29) $\frac{-10-9i}{6i}$

30) $\frac{-4+2i}{3i}$

31) $\frac{-3-6i}{4i}$

32) $\frac{-5+9i}{9i}$

33) $\frac{10-i}{-i}$

34) $\frac{10}{5i}$

35) $\frac{4i}{-10+i}$

36) $\frac{9i}{1-5i}$

37) $\frac{8}{-10+i}$

38) $\frac{4}{4+6i}$

39) $\frac{7}{10-7i}$

40) $\frac{9}{-8-6i}$

41) $\frac{5i}{-6-i}$

42) $\frac{8i}{6-7i}$

Answers - Square Roots

1) $7\sqrt{5}$
2) $5\sqrt{5}$
3) 6
4) 14
5) $2\sqrt{3}$
6) $6\sqrt{2}$
7) $6\sqrt{3}$
8) $20\sqrt{2}$
9) $48\sqrt{2}$
10) $56\sqrt{2}$
11) $-112\sqrt{2}$
12) $-21\sqrt{7}$
13) $8\sqrt{3n}$
14) $7\sqrt{7b}$
15) $14v$
16) $10n\sqrt{n}$
17) $6x\sqrt{7}$
18) $10a\sqrt{2a}$
19) $-10k^2$
20) $-20p^2\sqrt{7}$
21) $-56x^2$
22) $-16\sqrt{2n}$
23) $-30\sqrt{m}$
24) $32p\sqrt{7}$
25) $3xy\sqrt{5}$
26) $6b^2a\sqrt{2a}$
27) $4xy\sqrt{xy}$
28) $16a^2b\sqrt{2}$
29) $8x^2y^2\sqrt{5}$
30) $16m^2n\sqrt{2n}$
31) $24y\sqrt{5x}$
32) $56\sqrt{2mn}$
33) $35xy\sqrt{5y}$
34) $12xy\sqrt{2}$
35) $-12u\sqrt{5uv}$
36) $-30y^2x\sqrt{2x}$
37) $-48x^2z^2y\sqrt{5}$
38) $30a^2c\sqrt{2b}$
39) $8j^2\sqrt{5hk}$
40) $-4yz\sqrt{2xz}$
41) $-12p\sqrt{6mn}$
42) $-32p^2m\sqrt{2q}$

Answers - Higher Roots

1) $5\sqrt[3]{5}$
2) $5\sqrt[3]{3}$
3) $5\sqrt[3]{6}$
4) $5\sqrt[3]{2}$
5) $5\sqrt[3]{7}$
6) $2\sqrt[3]{3}$
7) $-8\sqrt[4]{6}$
8) $-16\sqrt[4]{3}$
9) $12\sqrt[4]{7}$
10) $6\sqrt[4]{3}$
11) $-2\sqrt[4]{7}$
12) $15\sqrt[4]{3}$
13) $3\sqrt[4]{8a^2}$
14) $2\sqrt[4]{4n^3}$
15) $2\sqrt[5]{7n^3}$
16) $-2\sqrt[5]{3x^4}$
17) $2p\sqrt[5]{7}$
18) $2x\sqrt[6]{4}$
19) $-6\sqrt[7]{7r}$
20) $-16b\sqrt[7]{3b}$
21) $4v^2\sqrt[3]{6v}$
22) $20a^2\sqrt[3]{2}$
23) $-28n^2\sqrt[3]{5}$
24) $-8n^2$
25) $-3xy\sqrt[3]{5x^2}$
26) $4uv\sqrt[3]{u^2}$
27) $-2xy\sqrt[3]{4xy}$
28) $10ab\sqrt[3]{ab^2}$
29) $4xy^2\sqrt[3]{4x}$
30) $3xy^2\sqrt[3]{7}$
31) $-21xy^2\sqrt[3]{3y}$
32) $-8y^2\sqrt[3]{7x^2y^2}$
33) $10v^2\sqrt[3]{3u^2v^2}$
34) $-40\sqrt[3]{6xy}$
35) $-12\sqrt[3]{3ab^2}$
36) $9y\sqrt[3]{5x}$
37) $-18m^2np^2\sqrt[3]{2m^2p}$
38) $-12mpq\sqrt[4]{5p^3}$
39) $18xy\sqrt[4]{8xy^3z^2}$
40) $-18b^2a\sqrt[4]{5ac}$
41) $14j^2k^2h\sqrt[4]{8h^2}$
42) $-18xz\sqrt[4]{4x^3yz^3}$

Answers - Add and Subtract Radicals

1) $6\sqrt{5}$
2) $-3\sqrt{6}-5\sqrt{3}$
3) $-3\sqrt{2}+6\sqrt{5}$
4) $-5\sqrt{6}-\sqrt{3}$
5) $-5\sqrt{6}$
6) $-3\sqrt{3}$
7) $3\sqrt{6}+5\sqrt{5}$
8) $-\sqrt{5}+\sqrt{3}$
9) $-8\sqrt{2}$
10) $-6\sqrt{6}+9\sqrt{3}$
11) $-3\sqrt{6}+\sqrt{3}$
12) $-2\sqrt{5}-6\sqrt{6}$
13) $-2\sqrt{2}$
14) $8\sqrt{5}-\sqrt{3}$
15) $5\sqrt{2}$
16) $-9\sqrt{3}$
17) $-3\sqrt{6}-\sqrt{3}$
18) $3\sqrt{2}+3\sqrt{6}$
19) $-12\sqrt{2}+2\sqrt{5}$
20) $-3\sqrt{2}$
21) $-4\sqrt{6}+4\sqrt{5}$
22) $-\sqrt{5}-3\sqrt{6}$
23) $8\sqrt{6}-9\sqrt{3}+4\sqrt{2}$
24) $-\sqrt{6}-10\sqrt{3}$
25) $2\sqrt[3]{2}$
26) $6\sqrt[3]{5}-3\sqrt[3]{3}$
27) $-\sqrt[4]{3}$
28) $10\sqrt[4]{4}$
29) $\sqrt[4]{2}-3\sqrt[4]{3}$
30) $5\sqrt[4]{6}+2\sqrt[4]{4}$
31) $3\sqrt[4]{4}$
32) $-6\sqrt[4]{3}+2\sqrt[4]{6}$
33) $2\sqrt[4]{2}+\sqrt[4]{3}+6\sqrt[4]{4}$
34) $-2\sqrt[4]{3}-9\sqrt[4]{5}-3\sqrt[4]{2}$
35) $\sqrt[5]{6}-6\sqrt[5]{2}$
36) $10\sqrt[5]{6}-9\sqrt[5]{5}$
37) $4\sqrt[5]{5}-4\sqrt[5]{6}$
38) $-11\sqrt[7]{2}-2\sqrt[7]{5}$
39) $-4\sqrt[6]{4}-6\sqrt[6]{5}-4\sqrt[6]{2}$
40) $\sqrt[7]{3}-6\sqrt[7]{6}+3\sqrt[7]{5}$

Answers - Multiply and Divide Radicals

1) $-48\sqrt{5}$
2) $-25\sqrt{6}$
3) $6m\sqrt{5}$
4) $-25r^2\sqrt{2r}$
5) $2x^2\sqrt[3]{x}$
6) $6a^2\sqrt[3]{5a}$
7) $2\sqrt{3}+2\sqrt{6}$
8) $5\sqrt{2}+2\sqrt{5}$
9) $-45\sqrt{5}-10\sqrt{15}$
10) $25\sqrt{6r}+20r^2\sqrt{15}$
11) $25n\sqrt{10}+10\sqrt{5}$
12) $5\sqrt{3}-9\sqrt{5}$
13) $-2-4\sqrt{2}$
14) $16-9\sqrt{3}$
15) $15-11\sqrt{5}$
16) $30+8\sqrt{3}+5\sqrt{15}+4\sqrt{5}$

17) $6a+a\sqrt{10}+6a\sqrt{6}+2a\sqrt{15}$

18) $-4p\sqrt{10}+50\sqrt{p}$

19) $63+32\sqrt{3}$

20) $-10\sqrt{m}+25\sqrt{2}+\sqrt{2m}-5$

21) $\frac{\sqrt{3}}{25}$

22) $\frac{\sqrt{15}}{4}$

23) $\frac{1}{20}$

24) 2

25) $\frac{\sqrt{15}}{3}$

26) $\frac{\sqrt{10}}{15}$

27) $\frac{4\sqrt{3}}{9}$

28) $\frac{4\sqrt{5}}{5}$

29) $\frac{5\sqrt{3xy}}{12y^2}$

30) $\frac{4\sqrt{3x}}{15y^2x}$

31) $\frac{\sqrt{6p}}{3}$

32) $\frac{2\sqrt{5n}}{5}$

33) $\frac{\sqrt[3]{10}}{5}$

34) $\frac{\sqrt[3]{15}}{4}$

35) $\frac{\sqrt[3]{10}}{8}$

36) $\frac{\sqrt[4]{8}}{8}$

37) $\frac{5\sqrt[4]{10r^2}}{2}$

38) $\frac{\sqrt[4]{4n^2}}{mn}$

Answers - Rationalize Denominators

1) $\frac{4+2\sqrt{3}}{3}$

2) $\frac{-4+\sqrt{3}}{12}$

3) $\frac{2+\sqrt{3}}{5}$

4) $\frac{\sqrt{3}-1}{4}$

5) $\frac{2\sqrt{13}-5\sqrt{65}}{52}$

6) $\frac{\sqrt{85}+4\sqrt{17}}{68}$

7) $\frac{\sqrt{6}-9}{3}$

8) $\frac{\sqrt{30}-2\sqrt{3}}{18}$

9) $\frac{2\sqrt{5}+15p}{50}$

10) $\frac{5\sqrt{19}-x\sqrt{57}}{38}$

11) $\frac{3-4m\sqrt{6}}{30m}$

12) $\frac{6\sqrt{7r}-\sqrt{42r}}{14}$

13) $\frac{15\sqrt{5}-5\sqrt{2}}{43}$

14) $\frac{-5\sqrt{3}+20\sqrt{5}}{77}$

15) $\frac{10-2\sqrt{2}}{23}$

16) $\frac{2\sqrt{3}+\sqrt{2}}{2}$

17) $\frac{-12-9\sqrt{3}}{11}$

18) $-2\sqrt{2}-4$

19) $3-\sqrt{5}$

20) $\frac{\sqrt{5}-\sqrt{3}}{2}$

21) $1-\sqrt{2}$

22) $\frac{16\sqrt{3}+4\sqrt{5}}{43}$

23) $\frac{5}{n^2-5}$

24) $\frac{-5n\sqrt{3n}+n^3\ \sqrt{6n}}{25-2n^4}$

25) $\frac{4p}{3-5p^2}$

26) $\frac{5x^2+3x^2\sqrt{5x}}{5-9x}$

27) $\frac{20-4x\sqrt{5}}{25-5x^2}$

28) $\frac{2b}{-2b^2-1}$

29) $\frac{10-5r\sqrt{5r}}{4-5r^3}$

30) $\frac{5a\sqrt{3}-20}{3a^3-16a}$

31) $\frac{-25v+15\sqrt{v}}{25v^2-9v}$

32) $\frac{8\sqrt{2}-4\sqrt{n}}{8n-n^2}$

33) $\frac{24-4\sqrt{6}+9\sqrt{2}-3\sqrt{3}}{15}$

34) $\frac{5+5\sqrt{3}+4\sqrt{5}+4\sqrt{15}}{10}$

35) $\frac{-1+\sqrt{5}}{4}$

36) $\frac{2\sqrt{5}-5\sqrt{2}-10+5\sqrt{10}}{30}$

37) $\frac{-5\sqrt{2}+10-\sqrt{3}+\sqrt{6}}{5}$

38) $\frac{-4\sqrt{10}-16\sqrt{2}-2\sqrt{15}-8\sqrt{3}}{11}$

39) $\frac{8+3\sqrt{6}}{10}$

40) $\frac{4\sqrt{3}+\sqrt{6}-4\sqrt{5}-\sqrt{10}}{14}$

41) $\frac{4\sqrt{3}-\sqrt{15}-4\sqrt{2}+\sqrt{10}}{11}$

42) $\frac{-2\sqrt{10}+6\sqrt{6}+\sqrt{15}-9}{22}$

43) $\frac{20-8x\sqrt{5x}+10x\sqrt{2}-4x^2\sqrt{10x}}{25-20x^3}$

44) $\frac{12p^2+3p^4\sqrt{2}+4p\sqrt{5p}+p^3\sqrt{10p}}{16-2p^4}$

45) $\frac{10m\sqrt{3}+6m^2-5m^2\sqrt{2}-m^3\sqrt{6}}{25-3m^2}$

46) $\frac{-16v+20v\sqrt{v}+8v\sqrt{2}-10v\sqrt{2v}}{16-25v}$

47) $\frac{-2b-2b^3\sqrt{2}+5\sqrt{2b}+10b^2\sqrt{b}}{1-2b^4}$

48) $\frac{-4n-n\sqrt{3}}{-3-n^2}$

49) $\frac{8+10\sqrt{3x}-4\sqrt{2x}-5x\sqrt{6}}{16x-75x^2}$

50) $\frac{9a+15a^2\sqrt{3}+6a\sqrt{2}+10a^2\sqrt{6}}{9-75a^2}$

51) $\frac{3p-4p\sqrt{p}+\sqrt{p}}{p-p^2}$

52) $\frac{-8+4\sqrt{3}+2\sqrt{5x}-\sqrt{15x}}{x}$

Answers - Rational Exponents

1) $(\sqrt[5]{m})^3$

2) $\frac{1}{(\sqrt[4]{10r})^3}$

3) $(\sqrt{7x})^3$

4) $\frac{1}{(\sqrt[3]{6v})^4}$

5) $(6x)^{-\frac{3}{2}}$

6) $v^{\frac{1}{2}}$

7) $n^{-\frac{7}{4}}$

8) $(5a)^{\frac{1}{2}}$

9) 4

10) 2

11) 8

12) $\frac{1}{1000}$

13) $x^{\frac{4}{3}}y^{\frac{5}{2}}$

14) $\frac{4}{v^{\frac{1}{3}}}$

15) $\frac{1}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$

16) 1

17) $\frac{1}{3a^2}$

18) $\frac{y^{\frac{25}{12}}}{x^{\frac{5}{6}}}$

19) $u^2v^{\frac{11}{2}}$

20) 1

21) $y^{\frac{1}{2}}$

22) $\frac{v^2}{u^{\frac{7}{2}}}$

23) $\frac{b^{\frac{7}{4}}a^{\frac{3}{4}}}{3}$

24) $\frac{2y^{\frac{17}{6}}}{x^{\frac{7}{4}}}$

25) $\frac{3y^{\frac{1}{12}}}{2}$

26) $\frac{a^{\frac{3}{2}}}{2b^{\frac{1}{4}}}$

27) $\frac{m^{\frac{35}{8}}}{n^{\frac{7}{6}}}$

28) $\frac{1}{y^{\frac{5}{4}}x^{\frac{3}{2}}}$

29) $\frac{1}{n^{\frac{3}{4}}}$

30) $\frac{y^{\frac{1}{3}}}{x^{\frac{1}{4}}}$

31) $xy^{\frac{4}{3}}$

32) $\frac{x^{\frac{4}{3}}}{y^{\frac{10}{3}}}$

33) $\frac{u^{\frac{1}{2}}}{v^{\frac{3}{4}}}$

34) $x^{\frac{15}{4}}y^{\frac{17}{4}}$

Answers - Mixed Index

1) $\sqrt[4]{4x^2y^3}$

2) $\sqrt{3xy^3}$

3) $\sqrt[6]{8x^2y^3z^4}$

4) $\sqrt{\frac{5}{4x}}$

5) $\frac{\sqrt[3]{36xy}}{3y}$

6) $\sqrt[5]{x^3y^4z^2}$

7) $\sqrt[4]{x^2y^3}$

8) $\sqrt[5]{8x^4y^2}$

9) $\sqrt[4]{x^3y^2z}$

10) $\sqrt{5y}$

11) $\sqrt[3]{2xy^2}$

12) $\sqrt[4]{3x^2y^3}$

13) $\sqrt[6]{5400}$

14) $\sqrt[12]{300125}$

15) $\sqrt[6]{49x^3y^2}$

16) $\sqrt[15]{27y^5z^5}$

17) $\sqrt[6]{x^3(x-2)^2}$

18) $\sqrt[4]{3x(y+4)^2}$

19) $\sqrt[10]{x^9y^7}$

20) $\sqrt[10]{4a^9b^9}$

21) $\sqrt[12]{x^{11}y^{10}}$

22) $\sqrt[20]{a^{18}b^{17}}$

23) $\sqrt[20]{a^{18}b^{17}c^{14}}$

24) $\sqrt[30]{x^{22}y^{11}z^{27}}$

25) $a\sqrt[4]{a}$

26) $x\sqrt{x}$

27) b$\sqrt[10]{b^9}$

28) $a\sqrt[12]{a^5}$

29) $xy\sqrt[6]{xy^5}$

30) $a\sqrt[10]{ab^7}$

31) $3a^2b\sqrt[4]{ab}$

32) $2xy^2\sqrt[6]{2x^5y}$

33) $3xy\sqrt[4]{xy}$

34) $a^2b^2c^2\sqrt[6]{a^2bc^2}$

35) $9a^2(b+1)\sqrt[6]{243a^5(b+1)^5}$

36) $4x(y+z)^3\sqrt[6]{2x(y+z)}$

37) $\sqrt[12]{a^5}$

38) $\sqrt[15]{x^7}$

39) $\sqrt[12]{x^2y^5}$

40) $\frac{\sqrt[15]{a^7b^{11}}}{b}$

41) $\sqrt[10]{ab^9c^7}$

42) $yz\sqrt[10]{xy^8z^3}$

43) $\sqrt[20]{(3x-1)^3}$

44) $\sqrt[12]{(2+5x)^5}$

45) $\sqrt[15]{(2x+1)^4}$

Answers - Complex Numbers

1) $11-4i$
2) $-4i$
3) $-3+9i$
4) $-1-6i$
5) $-3-13i$
6) $5-12i$
7) $-4-11i$
8) $-3-6i$
9) $-8-2i$
10) $13-8i$
11) 48
12) 24
13) 40
14) 32
15) -49
16) $28-21i$
17) $11+60i$
18) $-32-128i$
19) $80-10i$
20) $36-36i$
21) $27+38i$
22) $-28+76i$
23) $44+8i$
24) $16-18i$
25) $-3+11i$
26) $-1+13i$
27) $9i+5$
28) $\frac{-3i-2}{3}$
29) $\frac{10i-9}{6}$
30) $\frac{4i+2}{3}$
31) $\frac{3i-6}{4}$
32) $\frac{5i+9}{9}$
33) $10i+1$
34) $-2i$
35) $\frac{-40i+4}{101}$
36) $\frac{9i-45}{26}$
37) $\frac{56+48i}{85}$
38) $\frac{4-6i}{13}$
39) $\frac{70+49i}{149}$
40) $\frac{-36+27i}{50}$
41) $\frac{-30i-5}{37}$
42) $\frac{48i-56}{85}$