Beginning and Intermediate Algebra

Chapter 2: Graphing

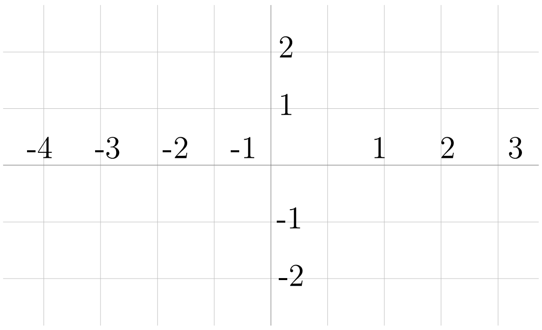
by Tyler Wallace

Chapter 2: Graphing Lines

2.1

Graphs - Points and Lines

Often, to get an idea of the behavior of an equation we will make a picture that represents the solutions to the equations. A **graph** is simplify a picture of the solutions to an equation. Before we spend much time on making a visual representation of an equation, we first have to understand the basis of graphing. Following is an example of what is called the coordinate plane.

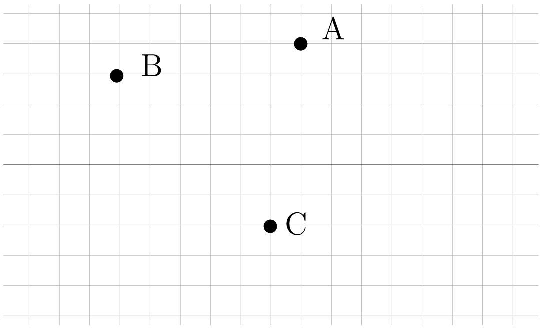


The plane is divided into four sections by a horizontal number line (x-axis) and a vertical number line (y-axis). Where the two lines meet in the center is called the origin. This center origin is where $x=0$ and $y=0$. As we move to the right the numbers count up from zero, representing $x=1,2,3....$

To the left the numbers count down from zero, representing $x=-1,-2,-3....$ Similarly, as we move up the number count up from zero, $y=1,2,3....$, and as we move down count down from zero, $y=-1,-2,-3$. We can put dots on the graph which we will call points. Each point has an "address" that defines its location. The first number will be the value on the $x-$axis or horizontal number line. This is the distance the point moves left/right from the origin. The second number will represent the value on the $y-$ axis or virtical number line. This is the distance the point moves up/down from the origin. The points are given as an ordered pair (x,y). The following example finds the address or coordinate pair for each of several points on the coordinate plane.

Example 1.

Give the coordinates of each point.



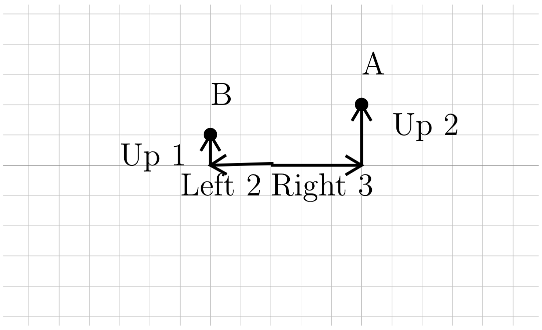
Tracing from the origin, point A is right 1, up 4. This becomes A(1, 4). Point B is left 5, up 3. Left is backwards or negative so we have B(-5, 3). C is stright down 2 units. There is no left or right. This means we go right zero so the point is C$(0,-2)$.

$$A(1,4), B(-5,3), C(0,-2) \qquad \text{Our Solution}$$

Just as we can give the coordinates for a set of points, we can take a set of points and plot them on the plane.

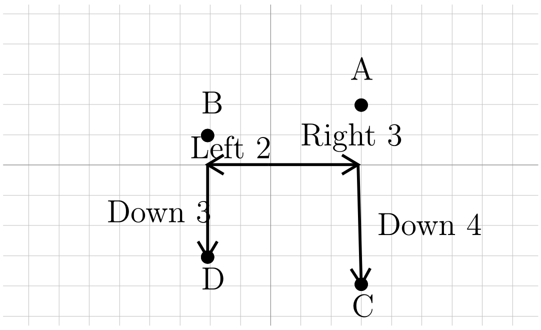
Example 2.

Graph the points $A(3, 2)$, $B(-2, 1)$, $C(3, -4)$, $D(-2, -3)$, $E(-3, 0)$, $F(0, 2)$, $G(0,0)$



The first point, A is at (3, 2) this means $x=3$ (right 3) and $y=2$ (up 2). Following these instructions, starting from the origin, we get our point.

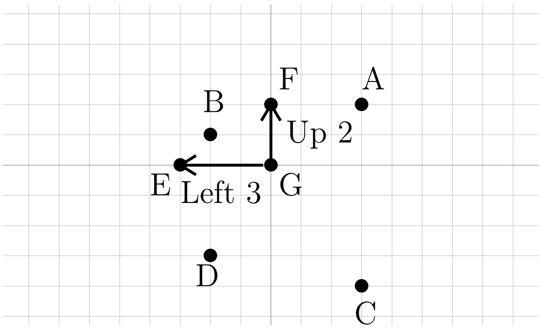
The second point, $B(-2, 1)$, is left 2 (negatives moves backwards), up 1. This is also illustrated on the graph.



The third point, $C(3, -4)$ is right 3, down 4 (negative moves backwards).

The fourth point, D $(-2, -3)$ is left 2, down 3 (both negative, both move backwards)

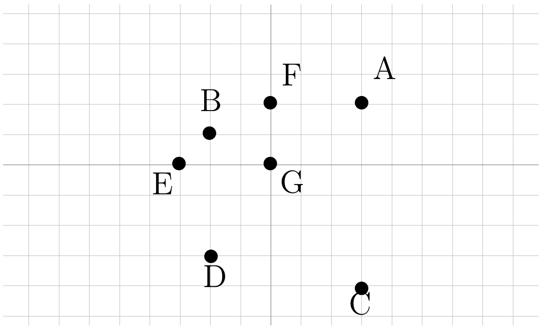
The last three points have zeros in them. We still treat these points just like the other points. If there is a zero there is just no movement.



Next is $E(-3,0)$. This is left 3 (negative is backwards), and up zero, right on the $x-$axis.

Then is F (0, 2). This is right zero, and up two, right on the $y-$axis.

Finally is $G(0, 0)$. This point has no movement. Thus the point is right on the origin.



Our Solution

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $y = 2x - 3$. We may be interested in what type of solution are possible in this equation. We can visualize the solution by making a graph of possible x and y combinations that make this equation a true statement. We will have to start by finding possible x and y combinations. We will do this using a table of values.

Example 3.

Graph $y = 2x - 3$ We make a table of values

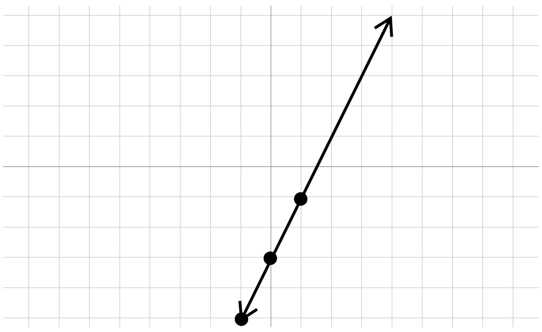
x	y
-1	
0	
1	

We will test three values for x. Any three can be used

x	y
-1	-5
0	-3
1	-1

Evaluate each by replacing x with the given value
$x = -1; y = 2(-1) - 3 = -2 - 3 = -5$
$x = 0; y = 2(0) - 3 = 0 - 3 = -3$
$x = 1; y = 2(1) - 3 = 2 - 3 = -1$

$(-1, -5), (0, -3), (1, -1)$ These then become the points to graph on our equation



Plot each point.

Once the point are on the graph, connect the dots to make a line.

The graph is our solution

What this line tells us is that any point on the line will work in the equation $y = 2x - 3$. For example, notice the graph also goes through the point $(2, 1)$. If we use $x = 2$, we should get $y = 1$. Sure enough, $y = 2(2) - 3 = 4 - 3 = 1$, just as the graph suggests. Thus we have the line is a picture of all the solutions for $y = 2x - 3$. We can use this table of values method to draw a graph of any linear equation.

Example 4.

Graph $2x - 3y = 6$ We will use a table of values

x	y
-3	
0	
3	

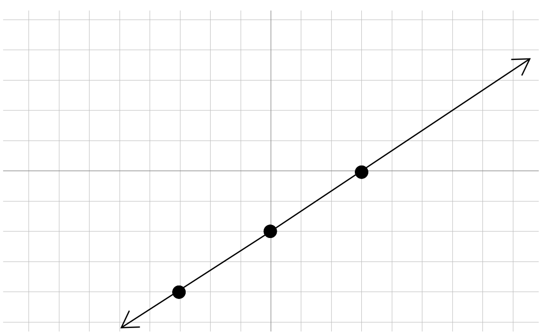
We will test three values for x. Any three can be used.

$2(-3)-3y=6$	Substitute each value in for x and solve for y
$-6-3y=6$	Start with $x=-3$, multiply first
$\underline{+6 \qquad +6}$	Add 6 to both sides
$-3y=12$	Divide both sides by -3
$\overline{-3} \quad \overline{-3}$	
$y=-4$	Solution for y when $x=-3$, add this to table
$2(0)-3y=6$	Next $x=0$
$-3y=6$	Multiplying clears the constant term
$\overline{-3}\ \overline{-3}$	Divide each side by -3
$y=-2$	Solution for y when $x=0$, add this to table
$2(3)-3y=6$	Next $x=3$
$6-3y=6$	Multiply
$\underline{-6 \qquad -6}$	Subtract 9 from both sides
$-3y=0$	Divide each side by -3
$\overline{-3} \quad \overline{-3}$	
$y=0$	Solution for y when $x=-3$, add this to table

x	y
-3	-4
0	-2
3	0

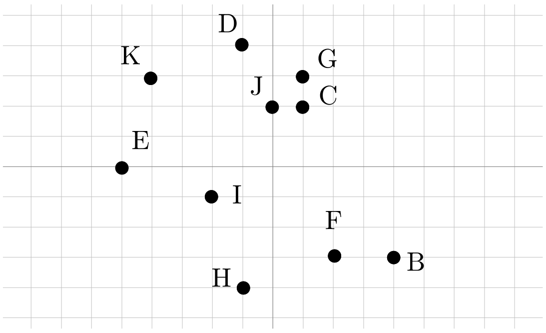
Our completed table.

$(-3,-4),(0,2),(3,0)$ Table becomes points to graph



Practice - Points and Lines

State the coordinates of each point.

1) 

Plot each point.

2) L$(-5,5)$ K$(1,0)$ $J(-3,4)$

I$(-3,0)$ $H(-4,2)$ G(4, -2)

$F(-2,-2)$ E $(3,-2)$ D(0, 3)

C (0, 4)

Sketch the graph of each line.

3) $y=-\frac{1}{4}x-3$

4) $y=x-1$

5) $y=-\frac{5}{4}x-4$

6) $y=-\frac{3}{5}x+1$

7) $y=-4x+2$

8) $y=\frac{5}{3}x+4$

9) $y=\frac{3}{2}x-5$

10) $y=-x-2$

11) $y=-\frac{4}{5}x-3$

12) $y=\frac{1}{2}x$

13) $x+5y=-15$

14) $8x-y=5$

15) $4x+y=5$

16) $3x+4y=16$

17) $2x-y=2$

18) $7x+3y=-12$

19) $x+y=-1$

20) $3x+4y=8$

21) $x-y=-3$

22) $9x-y=-4$

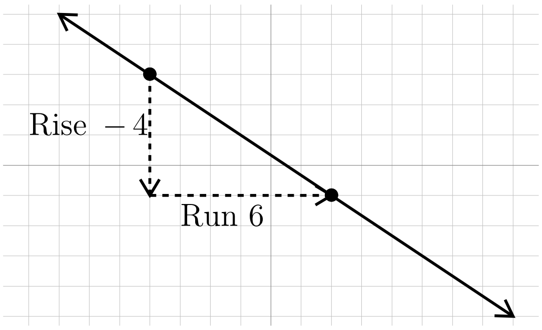
2.2

Graphing - Slope

As we graph lines, we will want to be able to identify differernt properties of the lines we graph. One of the most important properties of a line is its slope. **Slope** is a measure of steepness. A line with a large slope, such as 25, is very steep. A line with a small slope, such as $\frac{1}{10}$ is very flat. We will also use slope to describe the direction of the line. A line that goes up from left to right will have a positive slope and a line that goes down from left to right will have a negative slope.

As we measure steepness we are interested in how fast the line rises compared to how far the line runs. For this reason we will describe slope as the fraction $\frac{\text{rise}}{\text{run}}$. Rise would be a virtical change, or a change in the y-values. Run would be a horizontal change, or a change in the x-values. So another way to describe slope would be the fraction $\frac{\text{change in } y}{\text{change in } x}$. It turns out that if we have a graph we can draw virtical and horiztonal lines from one point to another to make what is called a slope triangle. The sides of the slope triangle give us our slope. The following examples show graphs that we find the slope of using this idea.

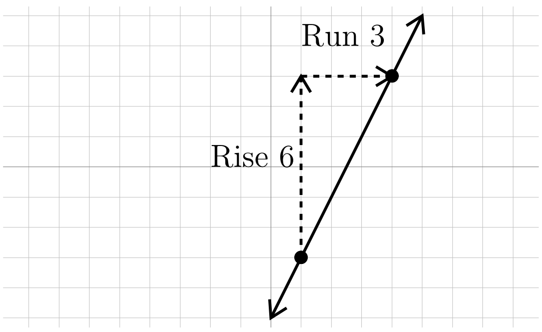
Example 5.



To find the slope of this line we will consider the rise, or verticle change and the run or horizontal change. Drawing these lines in makes a slope triangle that we can use to count from one point to the next the graph goes down 4, right 6. This is rise -4, run 6. As a fraction it would be, $\frac{-4}{6}$. Reduce the fraction to get $-\frac{2}{3}$.

$$-\frac{2}{3} \quad \text{Our Solution}$$

Example 6.



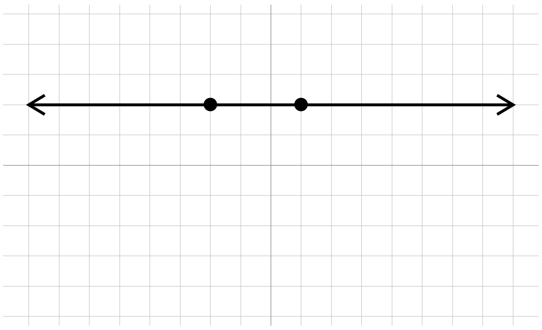
To find the slope of this line, the rise is up 6, the run is right 3. Our slope is then written as a fraction, $\frac{\text{rise}}{\text{run}}$ or $\frac{6}{3}$.

This fraction reduces to 2. This will be our slope.

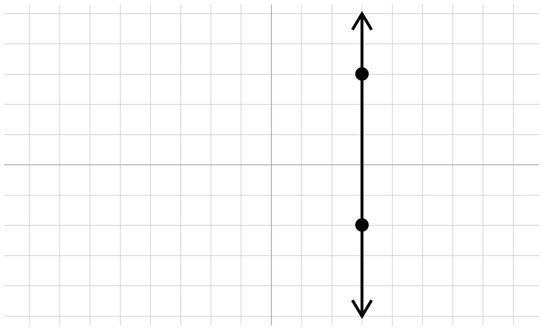
2 Our Solution

There are two special lines that have unique slopes that we need to be aware of. They are illustrated in the following example.

Example 7.



In this graph there is no rise, but the run is 3 units. This slope becomes $\frac{0}{3} = 0$. This line, and all horizontal lines have *a* zero slope.



This line has a rise of 5, but no run. The slope becomes $\frac{5}{0}$ = undefined. This line, and all virtical lines, have no slope.

As you can see there is a big difference between having a zero slope and having no slope or undefined slope. Remember, slope is a measure of steepness. The first slope is not steep at all, in fact it is flat. Therefore it has a zero slope. The second slope can't get any steeper. It is so steep that there is no number large enough to express how steep it is. This is an undefined slope.

We can find the slope of a line through two points without seeing the points on a graph. We can do this using a slope formula. If the rise is the change in y values, we can calculate this by subtracting the y values of a point. Similarly, if run is a the change in x values, we can calculate this by subtracting the x values of a point. In this way we get the following equation for slope.

$$\textbf{The slope of } \boldsymbol{a} \textbf{ line through } (\boldsymbol{x_1}, \boldsymbol{y_1}) \textbf{ and } (\boldsymbol{x_2}, \boldsymbol{y_2}) \textbf{ is } \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}}$$

When mathematicians began working with slope, it was called the modular slope. For this reason we often represent the slope with the variable m. Now we have the following for slope.

$$\textbf{Slope} = \boldsymbol{m} = \frac{\textbf{rise}}{\textbf{run}} = \frac{\textbf{change in } \boldsymbol{y}}{\textbf{change in } \boldsymbol{x}} = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}}$$

As we subtract the y values and the x values when calculating slope it is important we subtract them in the same order. This process is shown in the following examples.

Example 8.

$$
\begin{array}{rl}
\text{Find the slope between } (-4,3) \text{ and } (2,-9) & \text{Idenfity } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-9-3}{2-(-4)} & \text{Simplify} \\
m = \dfrac{-12}{6} & \text{Reduce} \\
m = -2 & \text{Our Solution}
\end{array}
$$

Example 9.

$$
\begin{array}{rl}
\text{Find the slope between } (4,\quad 6) \text{ and } (2,-1) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-1-6}{2-4} & \text{Simplify} \\
m = \dfrac{-5}{-2} & \text{Reduce, dividing by } -1 \\
m = \dfrac{5}{2} & \text{Our Solution}
\end{array}
$$

We may come up against a problem that has a zero slope (horiztonal line) or no slope (virticle line) just as with using the graphs.

Example 10.

$$
\begin{array}{rl}
\text{Find the slope between } (-4,-1) \text{ and } (-4,-5) & \text{Idenfity } x_1, y_1, x_2, y_2 \\
(x_1\ , y_1) \text{ and } \quad (x_2,\ \ y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{-5-(-1)}{-4-(-4)} & \text{Simplify} \\
m = \dfrac{-4}{0} & \text{Can}'t \text{ divide by zero, undefined} \\
m = \text{no solpe} & \text{Our Solution}
\end{array}
$$

Example 11.

$$
\begin{array}{rl}
\text{Find the slope between } (3,1) \text{ and } (-2,1) & \text{Identify } x_1, y_1, x_2, y_2 \\
(x_1, y_1) \text{ and } (x_2, y_2) & \text{Use slope formula, } m = \dfrac{y_2 - y_1}{x_2 - x_1} \\
m = \dfrac{1-1}{-2-3} & \text{Simplify} \\
m = \dfrac{0}{-5} & \text{Reduce} \\
m = 0 & \text{Our Solution}
\end{array}
$$

Again, there is a big difference between no slope and a zero slope. Zero is an integer and it has a value, the slope of a flat horizontal line. No slope has no value, it is undefined, the slope of a virticle line.

Using the slope formula we can also find missing points if we know what the slope is. This is shown in the following two examples.

Example 12.

Find the value of x between the points $(2, y)$ and $(5, -1)$ with slope -3

$$\begin{array}{rl}
m = \dfrac{y_2 - y_1}{x_2 - x_1} & \text{We will plug values into slope formula} \\
-3 = \dfrac{-1 - y}{5 - 2} & \text{Simplify} \\
-3 = \dfrac{-1 - y}{3} & \text{Multiply both sides by 3} \\
-3(3) = \dfrac{-1 - y}{3}(3) & \text{Simplify} \\
-9 = -1 - y & \text{Add 1 to both sides} \\
\underline{+1 \quad +1} & \\
-8 = -y & \text{Divide both sides by} -1 \\
\overline{-1} \quad \overline{-1} & \\
8 = y & \text{Our Solution}
\end{array}$$

Example 13.

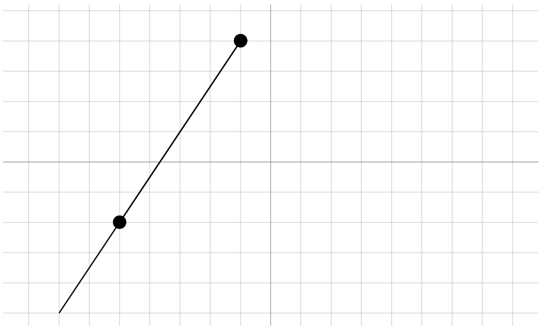
Find the value of x between the points $(2, -3)$ and $(x, 6)$ with slope $\frac{2}{5}$

$$\begin{array}{rl}
m = \dfrac{y_2 - y_1}{x_2 - x_1} & \text{We will plug values into slope formula} \\
\dfrac{2}{5} = \dfrac{6 - 2}{x - (-3)} & \text{Simplify} \\
\dfrac{2}{5} = \dfrac{4}{x + 3} & \text{Multiply both sides by } (x + 3) \\
\dfrac{2}{5}(x + 3) = 4 & \text{Multiply by 5 to clear fraction} \\
(5)\dfrac{2}{5}(x + 3) = 4(5) & \text{Simplify} \\
2(x + 3) = 20 & \text{Distribute} \\
2x + 6 = 20 & \text{Solve.} \\
\underline{-6 \quad -6} & \text{Subtract 6 from both sides} \\
2x = 14 & \text{Divide each side by 2} \\
\overline{2} \quad \overline{2} & \\
x = 7 & \text{Our Solution}
\end{array}$$

Practice - Slope

Find the slope of each line.

1)



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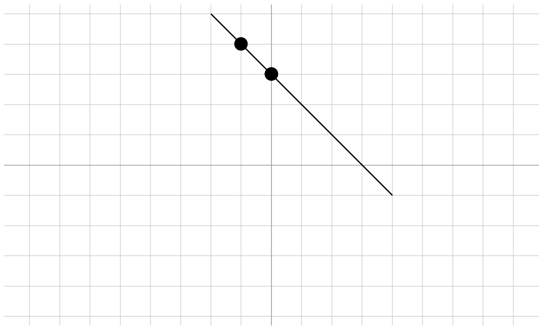

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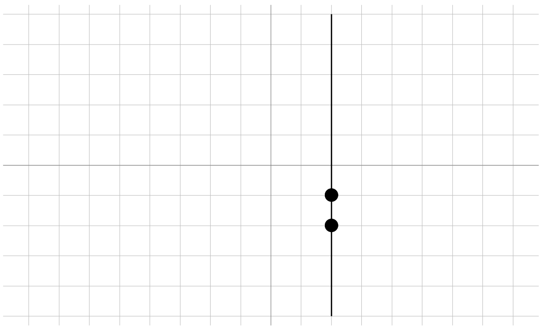

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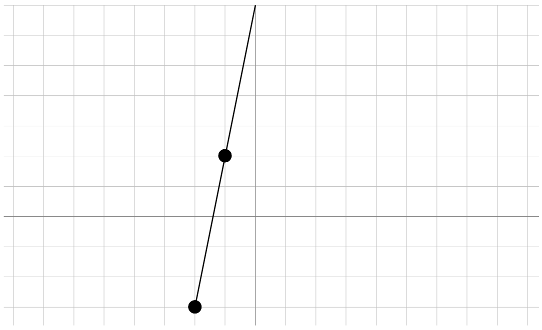

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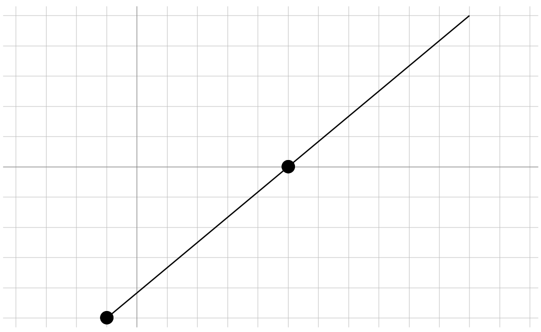

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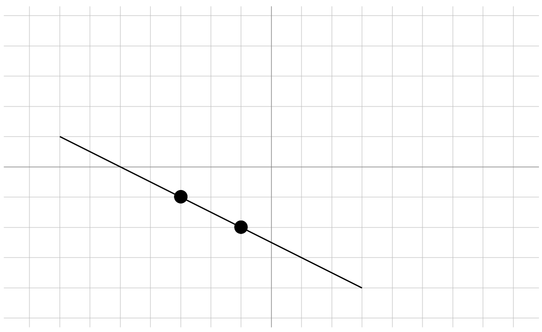

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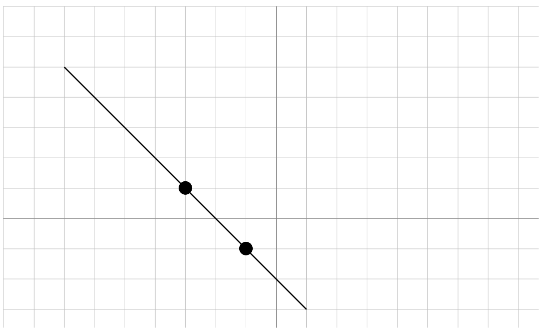

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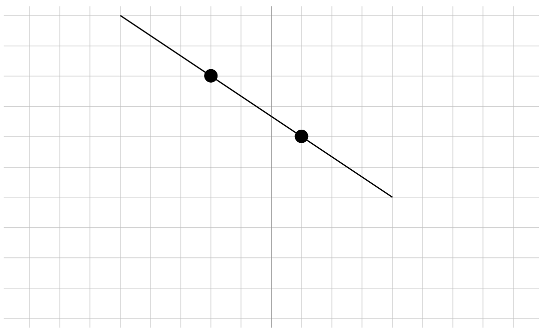

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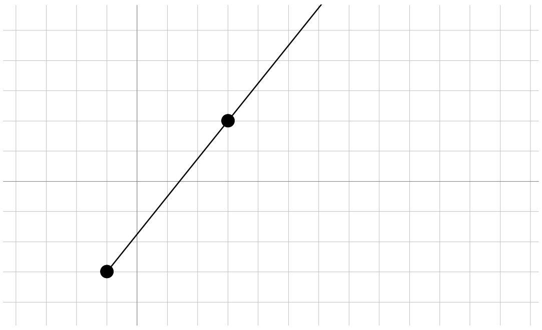
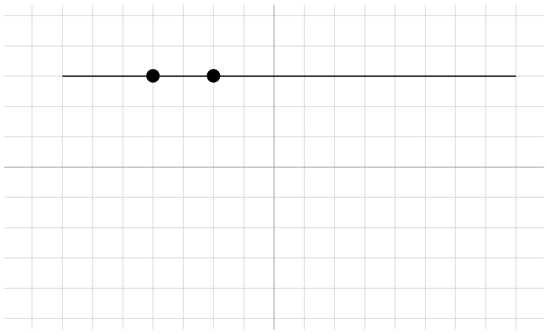




10)

Find the slope of the line through each pair of points.

11) $(-2, 10), (-2, -15)$

12) $(1, 2), (-6, -14)$

13) $(-15, 10), (16, -7)$

14) $(13, -2), (7, 7)$

15) $(10, 18), (-11, -10)$

16) $(-3, 6), (-20, 13)$

17) $(-16, -14), (11, -14)$

18) $(13, 15), (2, 10)$

19) $(-4, 14), (-16, 8)$

20) $(9, -6), (-7, -7)$

21) $(12, -19), (6, 14)$

22) $(-16, 2), (15, -10)$

23) $(-5, -10), (-5, 20)$

24) $(8, 11), (-3, -13)$

25) $(-17, 19), (10, -7)$

26) $(11, -2), (1, 17)$

27) $(7, -14), (-8, -9)$

28) $(-18, -5), (14, -3)$

29) $(-5, 7), (-18, 14)$

30) $(19, 15), (5, 11)$

Find the value of x or y so that the line through the points has the given slope.

31) $(2, 6)$ and $(x, 2)$; slope: $\frac{4}{7}$

32) $(8, y)$ and $(-2, 4)$; slope: $-\frac{1}{5}$

33) $(-3, -2)$ and $(x, 6)$; slope: $-\frac{8}{5}$

34) $(-2, y)$ and $(2, 4)$; slope: $\frac{1}{4}$

35) $(-8, y)$ and $(-1, 1)$; slope: $\frac{6}{7}$

36) $(x, -1)$ and $(-4, 6)$; slope: $-\frac{7}{10}$

37) $(x, -7)$ and $(-9, -9)$; slope: $\frac{2}{5}$

38) $(2, -5)$ and $(3, y)$; slope: 6

39) $(x, 5)$ and $(8, 0)$; slope: $-\frac{5}{6}$

40) $(6, 2)$ and $(x, 6)$; slope: $-\frac{4}{5}$

2.3

Graphing - Slope Intercept Form

When graphing a line we found one method we could use is to make a table of values. However, if we can identify some properties of the line, we may be able to make a graph much quicker and easier. One such method is finding the slope and the y-intercept of the equation. The slope can be represented by m and the y-intercept, where it crosses the axis and $x=0$, can be represented by $(0,b)$ where b is the value where the graph crosses the virtical y-axis. Any other point on the line can be represented by (x,y). Using this information we will look at the slope formula and solve the formula for y.

Example 14.

$$\begin{array}{rl} m,(0,b),(x,y) & \text{Using the slope formula gives:} \\ \dfrac{y-b}{x-0}=m & \text{Simplify} \\ \dfrac{y-b}{x}=m & \text{Multiply both sides by } x \\ y-b=mx & \text{Add } b \text{ to both sides} \\ \underline{+b\quad+b} & \\ y=mx+b & \text{Our Solution} \end{array}$$

This equation, $y=mx+b$ can be thought of as the equation of any line that as a slope of m and a y-intercept of b. This formula is known as the slope-intercept equation.

$$\textbf{Slope} - \textbf{Intercept Equation: } \boldsymbol{y=mx+b}$$

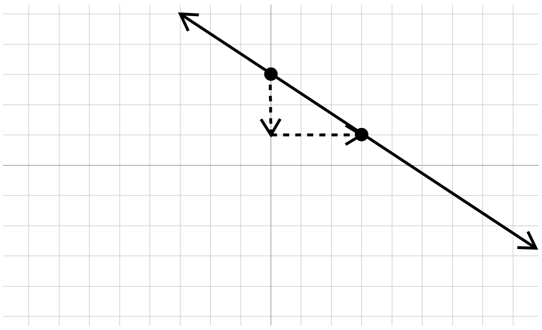
If we know the slope and the y-intercept we can easily find the equation that represents the line.

Example 15.

$$\begin{array}{rl} \text{Slope}=\dfrac{3}{4}, y-\text{intercept}=-3 & \text{Use the slope} - \text{intercept equation} \\ y=mx+b & m \text{ is the slope, } b \text{ is the } y-\text{intercept} \\ y=\dfrac{3}{4}x-3 & \text{Our Solution} \end{array}$$

We can also find the equation by looking at a graph and finding the slope and y-intercept.

Example 16.



Identify the point where the graph crosses the y-axis (0,3). This means the y-intercept is 3.

Idenfity one other point and draw a slope triangle to find the slope. The slope is $-\frac{2}{3}$

$y = mx + b$ Slope-intercept equation

$$y = -\frac{2}{3}x + 3 \quad \text{Our Solution}$$

We can also move the opposite direction, using the equation idenfity the slope and y-intercept and graph the equation from this information. However, it will be important for the equation to first be in slope intercept form. If it is not, we will have to solve it for y so we can identify the slope and the y-intercept.

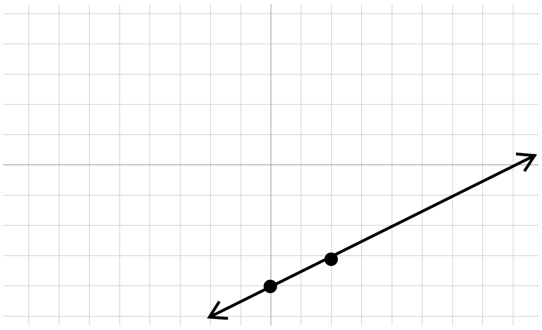
Example 17.

$$\begin{array}{rl} \text{Write in slope} - \text{intercept form: } 2x - 4y = 6 & \text{Solve for } y \\ -2x \qquad -2x & \text{Subtract } 2x \text{ from both sides} \\ -4y = -2x + 6 & \text{Put } x \text{ term first} \\ \overline{-4} \qquad \overline{-4}\ \overline{-4} & \text{Divide each term by } -4 \\ y = \frac{1}{2}x - \frac{3}{2} & \text{Our Solution} \end{array}$$

Once we have an equation in slope-intercept form we can graph it by first plotting the y-intercept, then using the slope, find a second point and connecting the dots.

Example 18.

$$\begin{array}{rl} \text{Graph } y = \frac{1}{2}x - 4 & \text{Recall the slope} - \text{intercept formula} \\ y = mx + b & \text{Idenfity the slope, } m\text{, and the } y - \text{intercept, } b \\ m = \frac{1}{2}, b = -4 & \text{Make the graph} \end{array}$$



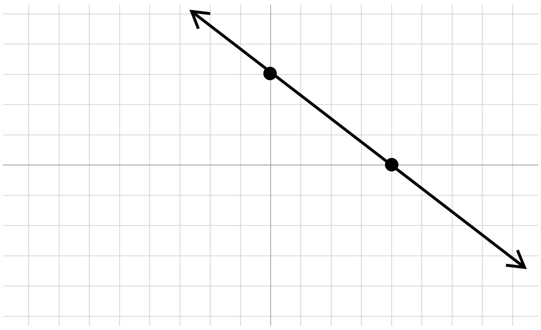
Starting with a point at the y-intercept of -4,

Then use the slope $\frac{\text{rise}}{\text{run}}$, so we will rise 1 unit and run 2 units to find the next point.

Once we have both points, connect the dots to get our graph.

Example 19.

$$
\begin{array}{rl}
\text{Graph } 3x+4y=12 & \text{Not in slope intercept form}\\
-3x \qquad -3x & \text{Subtract } 3x \text{ from both sides}\\
4y=-3x+12 & \text{Put the } x \text{ term first}\\
\overline{4} \qquad \overline{4} \quad \overline{4} & \text{Divide each term by } 4\\
y=-\dfrac{3}{4}x+3 & \text{Recall slope} - \text{intercept equation}\\
y=mx+b & \text{Idenfity } m \text{ and } b\\
m=-\dfrac{3}{4}, b=3 & \text{Make the graph}
\end{array}
$$



Starting with a point at the y-intercept of 3,

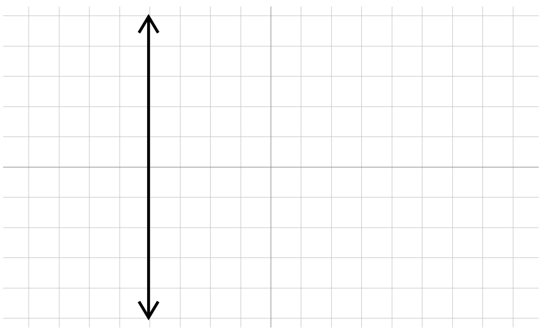
Then use the slope $\frac{\text{rise}}{\text{run}}$, but its negative so it will go downhill, so we will drop 3 units and run 4 units to find the next point.

Once we have both points, connect the dots to get our graph.

We want to be very careful not to confuse using slope to find the next point with use a coordinate such as $(4, -2)$ to find an individule point. Coordinates such as $(4, -2)$ start from the origin and move horiztonally first, and virtically second. Slope starts from a point on the line that could be anywhere on the graph. The numerator is the virticle change and the denominator is the horizontal change.

Lines with zero slope or no slope can make a problem seem very different. Zero slope, or horiztonal line, will simplify have a slope of zero which when multiplied by x gives zero. So the equation simply becomes $y=b$ or y is equal to the y-coordinate of the graph. If we have no slope, or a virticle line, the equation can't be written in slope intercept at all because the slope is undefined. There is no y in these equations. We will simplify make x equal to the x-coordinate of the graph.

Example 20.



Give the equation of the line in the graph.

Because we have a virticle line and no slope there is no slope-intercept equation we can use. Rather we make x equal to the x-coordinate of -4

$x=-4$ Our Solution

Practice - Slope-Intercept

Write the slope-intercept form of the equation of each line given the slope and the y-intercept.

1) Slope $= 2$, y-intercept $= 5$

2) Slope $= -6$, y-intercept $= 4$

3) Slope $= 1$, y-intercept $= -4$

4) Slope $= -1$, y-intercept $= -2$

5) Slope $= -\frac{3}{4}$, y-intercept $= -1$

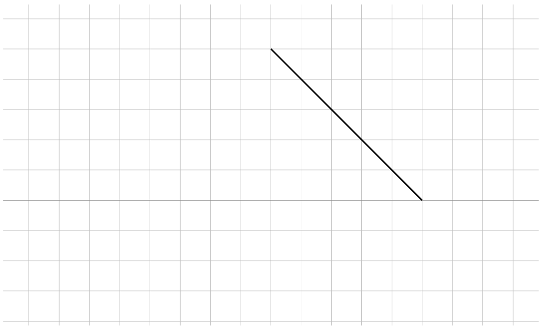
6) Slope $= -\frac{1}{4}$, y-intercept $= 3$

7) Slope $= \frac{1}{3}$, y-intercept $= 1$

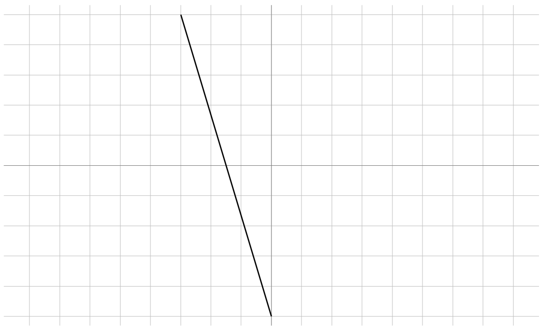
8) Slope $= \frac{2}{5}$, y-intercept $= 5$

Write the slope-intercept form of the equation of each line.

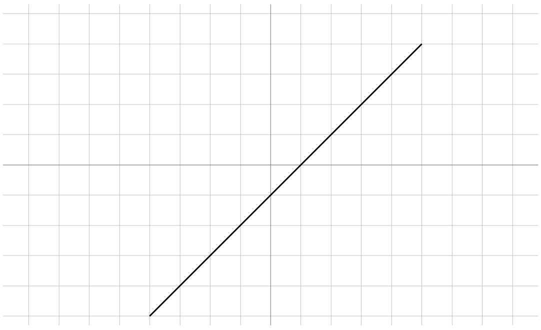
9)



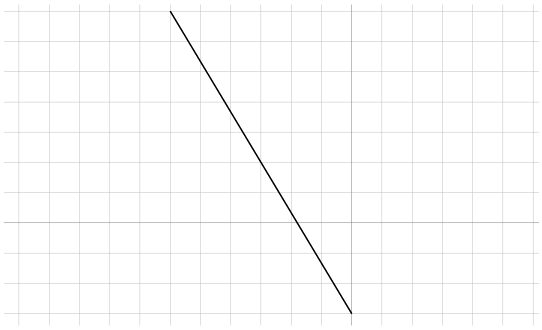
10)



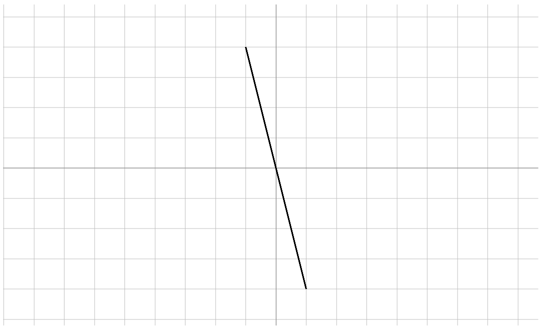
11)



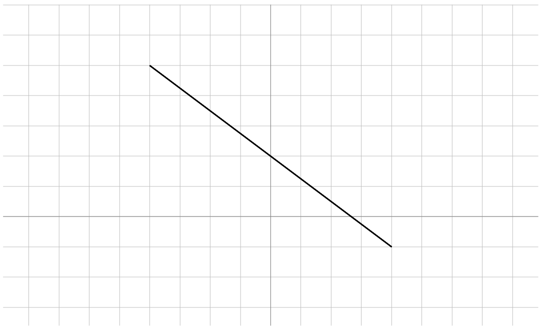
12)



13)



14)



15) $x + 10y = -37$

16) $x - 10y = 3$

17) $2x + y = -1$

18) $6x - 11y = -70$

19) $7x - 3y = 24$

20) $4x + 7y = 28$

21) $x = -8$

22) $x - 7y = -42$

23) $y - 4 = -(x + 5)$

24) $y - 5 = \frac{5}{2}(x - 2)$

25) $y - 4 = 4(x - 1)$

26) $y - 3 = -\frac{2}{3}(x + 3)$

27) $y + 5 = -4(x - 2)$

28) $0 = x - 4$

29) $y + 1 = \frac{1}{2}(x - 4)$

30) $y + 2 = \frac{6}{5}(x + 5)$

Sketch the graph of each line.

31) $y = \frac{1}{3}x + 4$

32) $y = -\frac{1}{5}x - 4$

33) $y = \frac{6}{5}x - 5$

34) $y = -\frac{3}{2}x - 1$

35) $y = \frac{3}{2}x$

36) $y = -\frac{3}{4}x + 1$

37) $x - y + 3 = 0$

38) $4x + 5 = 5y$

39) $-y - 4 + 3x = 0$

40) $-8 = 6x - 2y$

41) $-3y = -5x + 9$

42) $-3y = 3 - \frac{3}{2}x$

2.4

Graphing - Point Slope Form

The slope-intercept form has the advantage of being simple to remember and use, however it has one major disadvantage: we must know the y-intercept in order to use it! Generally we do not know the y-intercept, we only know one or more points (that are not the y-intercept). In these cases we can't use the slope intercept equation, so we will use a different more flexible formula. If we let the slope of an equation be m, and a specific point on the line be $(x_1,\, y_1)$, and any other point on the line be $(x,\, y)$. We can use the slope formula to make a second equation.

Example 21.

$$\begin{array}{rl} m, (x_1, y_1), (x, y) & \text{Recall slope formula} \\ \dfrac{y_2 - y_1}{x_2 - x_1} = m & \text{Plug in values} \\ \dfrac{y - y_1}{x - x_1} = m & \text{Multiply both sides by } (x - x_1) \\ y - y_1 = m(x - x_1) & \text{Our Solution} \end{array}$$

If we know the slope, m of an equation and any point on the line $(x_1,\, y_1)$ we can easily plug these values into the equation above which will be called the point-slope formula.

$$\textbf{Point} - \textbf{Slope Formula: } \boldsymbol{y - y_1 = m(x - x_1)}$$

Example 22.

Write the equation of the line through the point $(3, -4)$ with a slope of $\frac{3}{5}$.

$$\begin{array}{rl} y - y_1 = m(x - x_1) & \text{Plug values into point} - \text{slope formula} \\ y - (-4) = \dfrac{3}{5}(x - 3) & \text{Simplify signs} \end{array}$$

$$y+4=\frac{3}{5}(x-3) \quad \text{Our Solution}$$

Often, we will prefer final answers be written in slope intercept form. If the directions ask for the answer in slope-intercept form we will simplify distribute the slope, then solve for y.

Example 23.

Write the equation of the line through the point $(-6,2)$ with a slope of $-\frac{2}{3}$ in slope-intercept form.

$$\begin{aligned}
y-y_1 &= m(x-x_1) && \text{Plug values into point} - \text{slope formula}\\
y-2 &= -\frac{2}{3}(x-(-6)) && \text{Simplify signs}\\
y-2 &= -\frac{2}{3}(x+6) && \text{Distribute slope}\\
y-2 &= -\frac{2}{3}x-4 && \text{Solve for } y\\
\underline{+2} &\quad \underline{\quad +2} &&\\
y &= -\frac{2}{3}x-2 && \text{Our Solution}
\end{aligned}$$

An important thing to observe about the point slope formula is that the operation between the x's and y's is subtraction. This means when you simplify the signs you will have the opposite of the numbers in the point. We need to be very careful with signs as we use the point-slope formula.

In order to find the equation of a line we will always need to know the slope. If we don't know the slope to begin with we will have to do some work to find it first before we can get an equation.

Example 24.

Find the equation of the line through the points $(-2,5)$ and $(4,-3)$.

$$\begin{aligned}
m &= \frac{y_2-y_1}{x_2-x_1} && \text{First we must find the slope}\\
m = \frac{-3-5}{4-(-2)} &= \frac{-8}{6} = -\frac{4}{3} && \text{Plug values in slope formula and evaluate}\\
y-y_1 &= m(x-x_1) && \text{With slope and either point, use point} - \text{slope formula}
\end{aligned}$$

$$y-5=-\frac{4}{3}(x-(-2)) \quad \text{Simplify signs}$$
$$y-5=-\frac{4}{3}(x+2) \quad \text{Our Solution}$$

Example 25.

Find the equation of the line through the points $(-3,4)$ and $(-1,-2)$ in slope-intercept form.

$$\begin{aligned}
m&=\frac{y_2-y_1}{x_2-x_1} && \text{First we must find the slope}\\
m=\frac{-2-4}{-1-(-3)}&=\frac{-6}{2}=-3 && \text{Plug values in slope formula and evaluate}\\
y-y_1&=m(x-x_1) && \text{With slope and either point, point}-\text{slope formula}\\
y-4&=-3(x-(-3)) && \text{Simplify signs}\\
y-4&=-3(x+3) && \text{Distribute slope}\\
y-4&=-3x-9 && \text{Solve for } y\\
\underline{+4\quad}&\underline{\quad\quad +4} && \text{Add 4 to both sides}\\
y&=-3x-5 && \text{Our Solution}
\end{aligned}$$

Example 26.

Find the equation of the line through the points $(6,-2)$ and $(-4,1)$ in slope-intercept form.

$$\begin{aligned}
m&=\frac{y_2-y_1}{x_2-x_1} && \text{First we must find the slope}\\
m=\frac{1-(-2)}{-4-6}&=\frac{3}{-10}=-\frac{3}{10} && \text{Plug values into slope formula and evaluate}\\
y-y_1&=m(x-x_1) && \text{Use slope and either point, use point}-\text{slope formula}\\
y-(-2)&=-\frac{3}{10}(x-6) && \text{Simplify signs}\\
y+2&=-\frac{3}{10}(x-6) && \text{Distribute slope}\\
y+2&=-\frac{3}{10}x+\frac{9}{5} && \text{Solve for } y\text{. Subtract 2 from both sides}\\
\underline{-2\quad}&\underline{\quad\quad -\frac{10}{5}} && \text{Using } \frac{10}{5} \text{ on right so we have } a \text{ common denominator}\\
y&=-\frac{3}{10}x-\frac{1}{5} && \text{Our Solution}
\end{aligned}$$

Practice - Point Slope

Write the point-slope form of the equation of the line through the given point with the given slope.

1) through $(2,3)$, slope $=$ undefined

2) through $(1,2)$, slope $=$ undefined

3) through $(2,2)$, slope $=\frac{1}{2}$

4) through $(2,1)$, slope $=-\frac{1}{2}$

5) through $(-1,-5)$, slope $=9$

6) through $(2,-2)$, slope $=-2$

7) through $(-4,1)$, slope $=\frac{3}{4}$

8) through $(4,-3)$, slope $=-2$

9) through $(0,-2)$, slope $=-3$

10) through $(-1,1)$, slope $=4$

11) through $(0,-5)$, slope $=-\frac{1}{4}$

12) through $(0,2)$, slope $=-\frac{5}{4}$

13) through $(-5,-3)$, slope $=\frac{1}{5}$

14) through $(-1,-4)$, slope $=-\frac{2}{3}$

15) through $(-1,4)$, slope $=$

16) through $(1,-4)$, slope $=-\frac{3}{2}$

Write the slope-intercept form of the equation of the line through the given point with the given slope.

17) through: $(-1,-5)$, slope $=2$

18) through: $(2,-2)$, slope $=-2$

19) through: $(5,-1)$, slope $=-\frac{3}{5}$

20) through: $(-2,-2)$, slope $=-\frac{2}{3}$

21) through: $(-4,1)$, slope $=\frac{1}{2}$

22) through: $(4,-3)$, slope $=-\frac{7}{4}$

23) through: $(4,-2)$, slope $=-\frac{3}{2}$

24) through: $(-2,0)$, slope $=-\frac{5}{2}$

25) through: $(-5,-3)$, slope $=-\frac{2}{5}$

26) through: $(3,3)$, slope $=\frac{7}{3}$

27) through: $(2,-2)$, slope $=1$

28) through: $(-4,-3)$, slope $=0$

29) through: $(-3,4)$, slope=undefined

30) through: $(-2,-5)$, slope $=2$

31) through: $(-4,2)$, slope $=-\frac{1}{2}$

32) through: $(5,3)$, slope $=\frac{6}{5}$

Write the point-slope form of the equation of the line through the given points.

33) through: $(-4,3)$ and $(-3,1)$

34) through: $(1,3)$ and $(-3,3)$

35) through: $(5,1)$ and $(-3,0)$

36) through: $(-4,5)$ and $(4,4)$

37) through: $(-4,-2)$ and $(0,4)$

38) through: $(-4,1)$ and $(4,4)$

39) through: $(3,5)$ and $(-5,3)$

40) through: $(-1,-4)$ and $(-5,0)$

41) through: $(3,-3)$ and $(-4,5)$

42) through: $(-1,-5)$ and $(-5,-4)$

Write the slope-intercept form of the equation of the line through the given points.

43) through: $(-5,1)$ and $(-1,-2)$

44) through: $(-5,-1)$ and $(5,-2)$

45) through: $(-5,5)$ and $(2,-3)$

46) through: $(1,-1)$ and $(-5,-4)$

47) through: $(4,1)$ and $(1,4)$

48) through: $(0,1)$ and $(-3,0)$

49) through: $(0,2)$ and $(5,-3)$

50) through: $(0,2)$ and $(2,4)$

51) through: $(0,3)$ and $(-1,-1)$

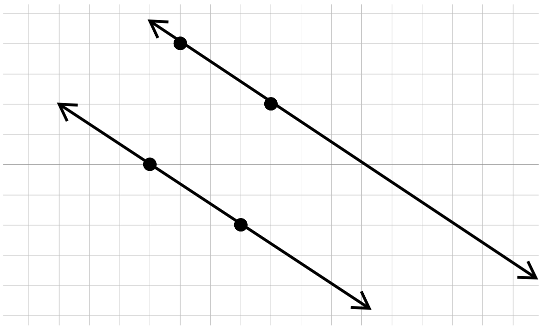
52) through: $(-2,0)$ and $(5,3)$

2.5

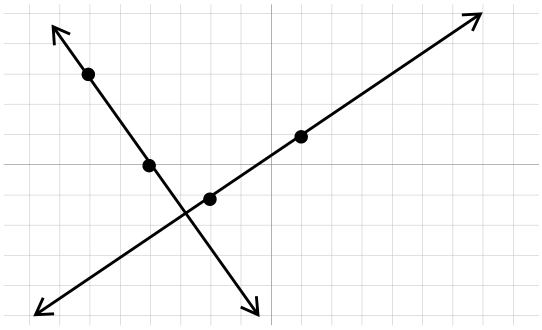
Graphing - Parallel and Perpendicular

There is an interesting connection between the slope of lines that are parallel and the slope of lines that are perpendicular (meet at a right angle). This is shown in the following example.

Example 27.



The above graph has two parallel lines. The slope of the top line is down 2, run 3, or $-\frac{2}{3}$. The slope of the bottom line is down 2, run 3 as well, or $-\frac{2}{3}$.



The above graph has two perpendicular lines. The slope of the flatter line is up 2, run 3 or $\frac{2}{3}$. The slope of the steeper line is down 3, run 1 or $-\frac{3}{2}$.

As the above graphs illustrate, parallel lines have the same slope and perpendicular lines have opposite (one positive, one negative) reciprocal (flipped fraction) slopes. We can use these properties to make conclusions about parallel and perpendicular lines.

Example 28.

Find the slope of a line parallel to $5y-2x=7$.

$$\begin{array}{rl} 5y-2x=7 & \text{To find the slope we will put equation in slope} - \text{intercept form} \\ \underline{+2x+2x} & \text{Add } 2x \text{ to both sides} \\ 5y=2x+7 & \text{Put } x \text{ term first} \\ \overline{5} \quad \overline{5} \quad \overline{5} & \text{Divide each term by 5} \\ y=\frac{2}{5}x+\frac{7}{5} & \text{The slope is the coefficient of } x \\ & \\ m=\frac{2}{5} & \text{Slope of first line. Parallel lines have the same slope} \\ & \\ m=\frac{2}{5} & \text{Our Solution} \end{array}$$

Example 29.

Find the slope of a line perpendicular to $3x - 4y = 2$

$$\begin{array}{rl}
3x - 4y = 2 & \text{To find slope we will put equation in slope} - \text{intercept form} \\
\underline{-3x \qquad -3x} & \text{Subtract } 3x \text{ from both sides} \\
-4y = -3x + 2 & \text{Put } x \text{ term first} \\
\overline{-4} \quad \overline{-4}\ \overline{-4} & \text{Divide each term by } -4 \\
y = \dfrac{3}{4}x - \dfrac{1}{2} & \text{The slope is the coefficient of } x \\
 & \\
m = \dfrac{3}{4} & \text{Slope of first lines. Perpendicular lines have opposite reciprocal slopes} \\
 & \\
m = -\dfrac{4}{3} & \text{Our Solution}
\end{array}$$

Once we have a slope, it is possible to find the complete equation of the second line if we know one point on the second line.

Example 30.

Find the equation of a line through $(4, -5)$ and parallel to $2x - 3y = 6$.

$$\begin{array}{rl}
2x - 3y = 6 & \text{We first need slope of parallel line} \\
\underline{-2x \qquad -2x} & \text{Subtract } 2x \text{ from each side} \\
-3y = -2x + 6 & \text{Put } x \text{ term first} \\
\overline{-3} \quad \overline{-3}\,\overline{-3} & \text{Divide each term by } -3 \\
y = \dfrac{2}{3}x - 2 & \text{Identify the slope, the coefficient of } x \\
 & \\
m = \dfrac{2}{3} & \text{Parallel lines have the same slope} \\
 & \\
m = \dfrac{2}{3} & \text{We will use this slope and our point } (4, -5) \\
 & \\
y - y_1 = m(x - x_1) & \text{Plug this information into point slope formula} \\
y - (-5) = \dfrac{2}{3}(x - 4) & \text{Simplify signs} \\
 & \\
y + 5 = \dfrac{2}{3}(x - 4) & \text{Our Solution}
\end{array}$$

Example 31.

Find the equation of the line through $(6, -9)$ perpendicular to $y = -\frac{3}{5}x + 4$ in slope-intercept form.

$$\begin{array}{rl}
y = -\dfrac{3}{5}x + 4 & \text{Identify the slope, coefficient of } x \\[2ex]
m = -\dfrac{3}{5} & \text{Perpendicular lines have opposite reciprocal slopes} \\[2ex]
m = \dfrac{5}{3} & \text{We will use this slope and our point } (6, -9) \\[2ex]
y - y_1 = m(x - x_1) & \text{Plug this information into point} - \text{slope formula} \\[2ex]
y - (-9) = \dfrac{5}{3}(x - 6) & \text{Simplify signs} \\[2ex]
y + 9 = \dfrac{5}{3}(x - 6) & \text{Distribute slope} \\[2ex]
y + 9 = \dfrac{5}{3}x - 10 & \text{Solve for } y \\
\underline{-9 \qquad\quad -9} & \text{Subtract 9 from both sides} \\
y = \dfrac{5}{3}x - 19 & \text{Our Solution}
\end{array}$$

Zero slopes and no slopes may seem like opposites (one is a horiztontal line, one is a virticle line). Because a horiztontal line is perpendicular to a virticle line we can say that no slope and zero slope are actually perpendicular slopes!

Example 32.

Find the equation of the line through $(3, 4)$ perpendicular to $x = -2$

$$\begin{array}{rl}
x = -2 & \text{This equation has no slope, } a \text{ virticle line} \\
\text{no slope} & \text{Perpendicular line then would have } a \text{ zero slope} \\
m = 0 & \text{Use this and our point } (3, 4) \\
y - y_1 = m(x - x_1) & \text{Plug this information into point} - \text{slope formula} \\
y - 4 = 0(x - 3) & \text{Distribute slope} \\
y - 4 = 0 & \text{Solve for } y \\
\underline{+4 + 4} & \text{Add 4 to each side} \\
y = 4 & \text{Our Solution}
\end{array}$$

Being aware that to be perpendicular to a virticle line means we have a horiztontal line through a y value of 4 we could have jumped from this point right to the solution, $y = 4$.

Practice - Solving Equations by Factoring

Find the slope of a line parallel to each given line.

1) $y=2x+4$

2) $y=-\frac{2}{3}x+5$

3) $y=4x-5$

4) $y=-\frac{10}{3}x-5$

5) $x-y=4$

6) $6x-5y=20$

7) $7x+y=-2$

8) $3x+4y=-8$

Find the slope of a line perpendicular to each given line.

9) $x=3$

10) $y=-\frac{1}{2}x-1$

11) $y=-\frac{1}{3}x$

12) $y=\frac{4}{5}x$

13) $x-3y=-6$

14) $3x-y=-3$

15) $x+2y=8$

16) $8x-3y=-9$

Write the point-slope form of the equation of the line described.

17) through: $(2,5)$, parallel to $x=0$

18) through: $(5,\ 2)$, parallel to $y=\frac{7}{5}x+4$

19) through: $(3,4)$, parallel to $y=\frac{9}{2}x-5$

20) through: $(1,-1)$, parallel to $y=-\frac{3}{4}x+3$

21) through: $(2,3)$, parallel to $y=\frac{7}{5}x+4$

22) through: $(-1,3)$, parallel to $y=-3x-1$

23) through: $(4,2)$, parallel to $x=0$

24) through: $(1,4)$, parallel to $y=\frac{7}{5}x+2$

25) through: $(1,\ -5)$, perpendicular to $y=x+1$

26) through: $(1,-2)$, perpendicular to $y=\frac{1}{2}x+1$

27) through: $(5,2)$, perpendicular to $y=-5x-3$

28) through: $(1,\ 3)$, perpendicular to $y=x+1$

29) through: $(4,2)$, perpendicular to $y=4x$

30) through: $(-3,-5)$, perpendicular to $y=-\frac{3}{7}x$

31) through: $(2,-2)$ perpendicular to $y=\frac{1}{3}x$

32) through: $(-2,5)$. perpendicular to $y=2x$

Write the slope-intercept form of the equation of the line described.

33) through: $(4,-3)$, parallel to $y=-2x$

34) through: $(-5,2)$, parallel to $y=\frac{3}{5}x$

35) through: $(-3,1)$, parallel to $y=-\frac{4}{3}x-1$

36) through: $(-4,0)$, parallel to $y=-\frac{5}{4}x+4$

37) through: $(-4,-1)$, parallel to $y=-\frac{1}{2}x+1$

38) through: $(2,3)$, parallel to $y=\frac{5}{2}x-1$

39) through: $(-2,-1)$, parallel to $y=-\frac{1}{2}x-2$

40) through: $(-5,-4)$, parallel to $y=\frac{3}{5}x-2$

41) through: $(4,3)$, perpendicular to $y=-x-1$

42) through: $(-3,-5)$, perpendicular to $y=-\frac{1}{2}x-2$

43) through: $(5,2)$, perpendicular to $x=0$

44) through: $(5,-1)$, perpendicular to $y=\frac{5}{2}x+5$

45) through: $(-2,5)$, perpendicular to $y=x-2$

46) through: $(2,-3)$, perpendicular to $y=\frac{2}{5}x-2$

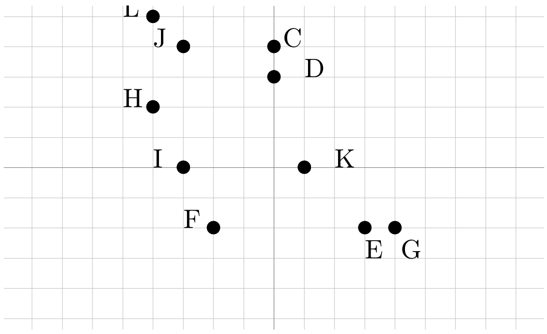
47) through: $(4,-3)$, perpendicular to $y=\frac{1}{2}x-3$

48) through: $(-4,1)$, perpendicular to $y=-\frac{4}{3}x-3$

Answers - Points and Lines

1) B(4, -3) C(1, 2) D($-1, 4$)
E($-5, 0$) F(2, -3) G(1, 3)
H($-1, -4$) I($-2, -1$) J(0, 2)
K($-4, 2$)

2)



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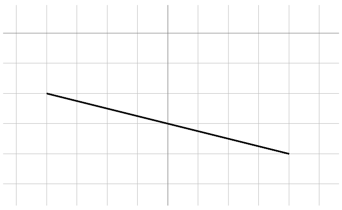

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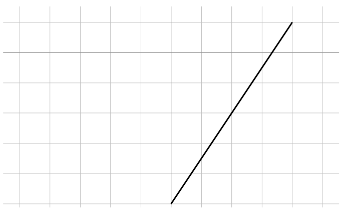

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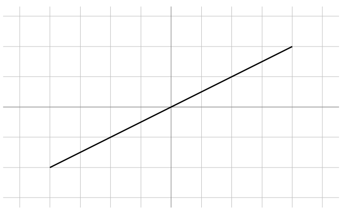

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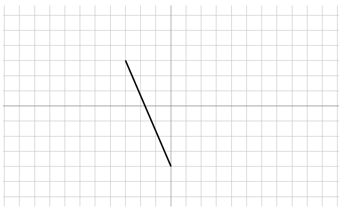

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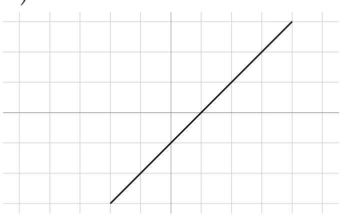

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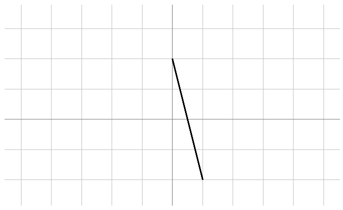
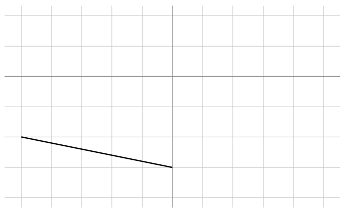


10)

13)



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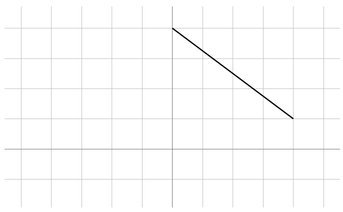
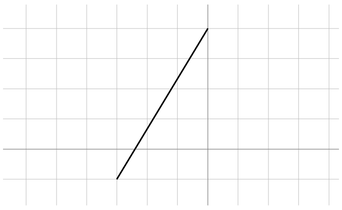


19)

8)



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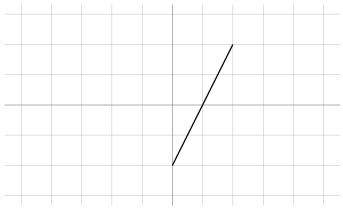

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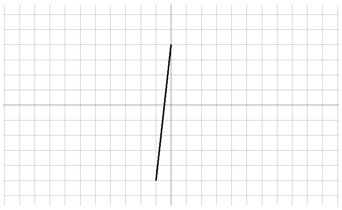

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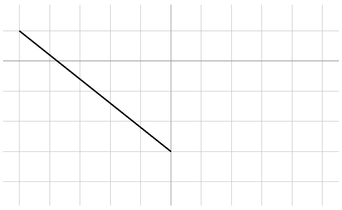

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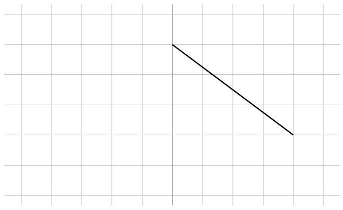

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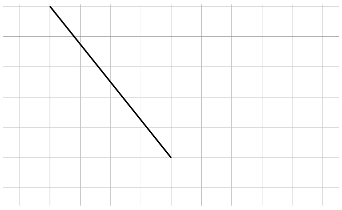

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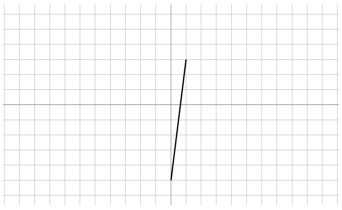


Answers - Slope

1) $\frac{3}{2}$

2) 5

3) Undefined

4) $-\frac{1}{2}$

5) $\frac{5}{6}$

6) $-\frac{2}{3}$

7) -1

8) $\frac{5}{4}$

9) -1

10) 0

11) Undefined

12) $\frac{16}{7}$

13) $-\frac{17}{31}$

14) $-\frac{3}{2}$

15) $\frac{4}{3}$

16) $-\frac{7}{17}$

17) 0

18) $\frac{5}{11}$

19) $\frac{1}{2}$

20) $\frac{1}{16}$

21) $-\frac{11}{2}$

22) $-\frac{12}{31}$

23) Undefined

24) $\frac{24}{11}$

25) $-\frac{26}{27}$

26) $-\frac{19}{10}$

27) $-\frac{1}{3}$

28) $\frac{1}{16}$

29) $-\frac{7}{13}$

30) $\frac{2}{7}$

31) -5

32) 2

33) -8

34) 3

35) -5

36) 6

37) -4

38) 1

39) 2

40) 1

Answers - Slope-Intercept

1) $y = 2x + 5$

2) $y = -6x + 4$

3) $y = x - 4$

4) $y = -x - 2$

5) $y = -\frac{3}{4}x - 1$

6) $y = -\frac{1}{4}x + 3$

7) $y = \frac{1}{3}x + 1$

8) $y = \frac{2}{5}x + 5$

9) $y = -x + 5$

10) $y = -\frac{7}{2}x - 5$

11) $y = x - 1$

12) $y = -\frac{5}{3}x - 3$

13) $y = -4x$

14) $y = -\frac{3}{4}x + 2$

15) $y = -\frac{1}{10}x - \frac{37}{10}$

16) $y = \frac{1}{10}x - \frac{3}{10}$

17) $y = -2x - 1$

18) $y = \frac{6}{11}x + \frac{70}{11}$

19) $y = \frac{7}{3}x - 8$

20) $y = -\frac{4}{7}x + 4$

21) $x = -8$

22) $y = \frac{1}{7}x + 6$

23) $y = -x - 1$

24) $y = \frac{5}{2}x$

25) $y = 4x$

26) $y = -\frac{2}{3}x + 1$

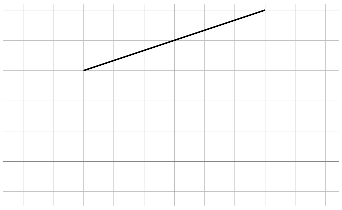
27) $y = -4x + 3$

28) $x = 4$

29) $y = -\frac{1}{2}x + 1$

30) $y = \frac{6}{5}x + 4$

31)



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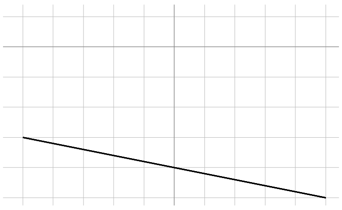

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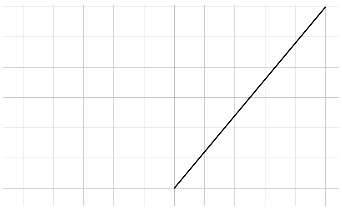

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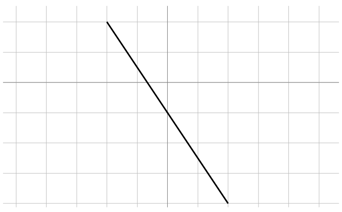

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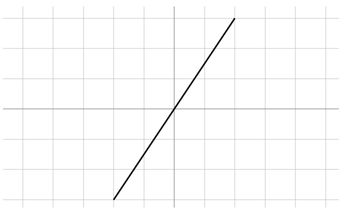

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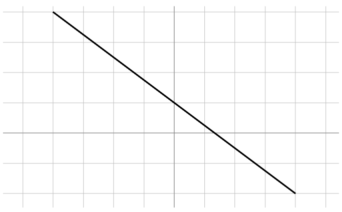

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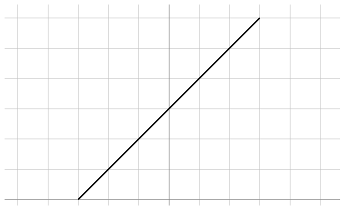

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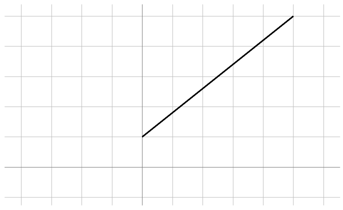
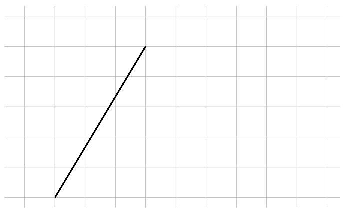


39)

40)

41)



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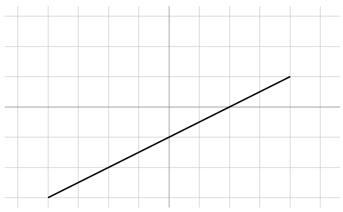

Answers - Point Slope

1) $0 = x - 2$

2) $0 = x - 1$

3) $y-2=\frac{1}{2}(x-2)$

4) $y-1=-\frac{1}{2}(x-2)$

5) $y+5=9(x+1)$

6) $y+2=-2(x-2)$

7) $y-1=\frac{3}{4}(x+4)$

8) $y+3=-2(x-4)$

9) $y+2=-3x$

10) $y-1=4(x+1)$

11) $y+5=-\frac{1}{4}x$

12) $y-2=-\frac{5}{4}x$

13) $y+3=\frac{1}{5}(x+5)$

14) $y+4=-\frac{2}{3}(x+1)$

15) $y-4=-\frac{5}{4}(x+1)$

16) $y+4=-\frac{3}{2}x(x-1)$

17) $y=2x-3$

18) $y=-2x+2$

19) $y=-\frac{3}{5}x+2$

20) $y=-\frac{2}{3}x-\frac{10}{3}$

21) $y=\frac{1}{2}x+3$

22) $y=-\frac{7}{4}x+4$

23) $y=-\frac{3}{2}x+4$

24) $y=-\frac{5}{2}x-5$

25) $y=-\frac{2}{5}x-5$

26) $y=\frac{7}{3}x-4$

27) $y=x-4$

28) $y=-3$

29) $x=-3$

30) $y=2x-1$

31) $y=-\frac{1}{2}x$

32) $y=\frac{6}{5}x-3$

33) $y-3=-2(x+4)$

34) $y-3=0$

35) $y-1=\frac{1}{8}(x-5)$

36) $y-5=-2(x+4)$

37) $y+2=\frac{3}{2}(x+4)$

38) $y-1=\frac{3}{8}(x+4)$

39) $y-5=\frac{1}{4}(x-3)$

40) $y+4=-(x+1)$

41) $y+3=-\frac{8}{7}(x-3)$

42) $y+5=-\frac{1}{4}(x+1)$

43) $y=-\frac{3}{4}x-\frac{11}{4}$

44) $y=-\frac{1}{10}x-\frac{3}{2}$

45) $y=-\frac{8}{7}x-\frac{5}{7}$

46) $y=\frac{1}{2}x-\frac{3}{2}$

47) $y=-x+5$

48) $y=\frac{1}{3}x+1$

49) $y=-x+2$

50) $y=x+2$

51) $y=4x+3$

52) $y=\frac{3}{7}x+\frac{6}{7}$

Answers - Parallel and Perpendicular Lines

1) 2

2) $-\frac{2}{3}$

3) 4

4) $-\frac{10}{3}$

5) 1

6) $\frac{6}{5}$

7) -7

8) $-\frac{3}{4}$

9) 0

10) 2

11) 3

12) $-\frac{5}{4}$

13) -3

14) $-\frac{1}{3}$

15) 2

16) $-\frac{3}{8}$

17) $0=x-2$

18) $y-2=\frac{7}{5}(x-5)$

19) $y-4=\frac{9}{2}(x-3)$

20) $y+1=-\frac{3}{4}(x-1)$

21) $y-3=\frac{7}{5}(x-2)$

22) $y-3=-3(x+1)$

23) $0=x-4$

24) $y-4=\frac{7}{5}(x-1)$

25) $y+5=-(x-1)$

26) $y+2=-2(x-1)$

27) $y-2=\frac{1}{5}(x-5)$

28) $y-3=-(x-1)$

29) $y-2=-\frac{1}{4}(x-4)$

30) $y+5=\frac{7}{3}(x+3)$

31) $y+2=-3(x-2)$

32) $y-5=-\frac{1}{2}(x+2)$

33) $y=-2x+5$

34) $y=\frac{3}{5}x+5$

35) $y=-\frac{4}{3}x-3$

36) $y=-\frac{5}{4}x-5$

37) $y=-\frac{1}{2}x-3$

38) $y=\frac{5}{2}x-2$

39) $y=-\frac{1}{2}x-2$

40) $y=\frac{3}{5}x-1$

41) $y=x-1$

42) $y=2x+1$

43) $y=2$

44) $y=-\frac{2}{5}x+1$

45) $y=-x+3$

46) $y=-\frac{5}{2}x+2$

47) $y=-2x+5$

48) $y=\frac{3}{4}x+4$