# Beginning and Intermediate Algebra

# Chapter 10: Functions



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# Chapter 10: Functions

10.1

## Functions - Function Notation

There are many different types of equations that we can work with in algebra. An equation gives the relationship between variables and numbers. Examples of several relationships are below:

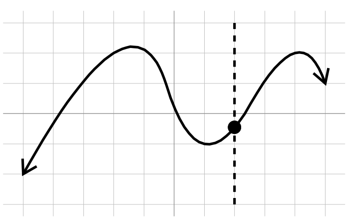
$$\frac{(x-3)^2}{9}-\frac{(y+2)^2}{4}=1 \quad \text{and} \quad y=x^2-2x+7 \quad \text{and} \quad \sqrt{y+x}-7=xy$$

There is a speical classification of relationships known as functions. **Functions** have at most one output for any input. Generally $x$ is the variable that we plug into an equation and evaluate to find $y$. For this reason $x$ is considered an input variable and $y$ is considered an output variable. This means the definition of a function, in terms of equations in $x$ and $y$ could be said, for any $x$ value there is at most one $y$ value that corresponds with it.

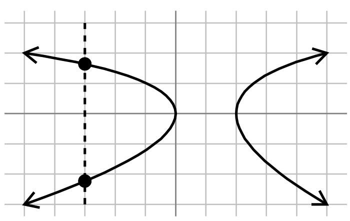
A great way to visulaize this definition is to look at the graphs of a few relationships. Because $x$ values are virticle lines we will draw a virticle line through the graph. If the virticle line crosses the graph more than once, that means we have to many possible $y$ values. If the graph crosses the graph only once, then we say the relationship is a function.

**Example 1.**

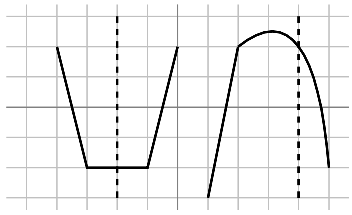
Which of the following graphs are graphs of functions?



Drawing a virticle line through this graph will only cross the graph once, it is a function.



Drawing a virticle line through this graph will cross the graph twice, once at top and once at bottom. This is not a function.



Drawing a virticle line through this graph will cross the graph only once, it is a function.

We can look at the above idea in an algebraic method by taking a relationship and solving it for $y$. If we have only one solution then it is a function.

**Example 2.**

$$\begin{array}{rl} \text{Is } 3x^2 - y = 5 \text{ a function?} & \text{Solve the relation for } y \\ \underline{-3x^2 \quad -3x^2} & \text{Subtract } 3x^2 \text{ from both sides} \\ -y = -3x^2 + 5 & \text{Divide each term by } -1 \\ \overline{-1} \quad\quad \overline{-1} \quad \overline{-1} & \\ y = 3x^2 - 5 & \text{Only one solution for } y. \\ \text{Yes!} & \text{It is } a \text{ function} \end{array}$$

**Example 3.**

$$\begin{array}{rl} \text{Is } y^2 - x = 5 \text{ a function?} & \text{Solve the relation for } y \\ +x + x & \text{Add } x \text{ to both sides} \\ y^2 = x + 5 & \text{Square root of both sdies} \\ \sqrt{y^2} = \pm\sqrt{x+5} & \text{Simplify} \\ y = \pm\sqrt{x+5} & \text{Two solutions for } y \text{ (one } +\text{, one } -\text{)} \\ \text{No!} & \text{Not } a \text{ function} \end{array}$$

Once we know we have a function, often we will change the notation used to emphasis the fact that it is a function. Instead of writting $y =$ , we will use functio notation which can be written $f(x) =$. We read this notation "$f$ of $x$". So for the above example that was a function, instead of writing $y = 3x^2 - 5$, we could have written $f(x) = 3x^2 - 5$. It is important to point out that $f(x)$ does not mean $f$ times $x$, it is mearly a notation that names the function with the first letter (function $f$) and then in parenthesis we are given information about what variables are in the function (variable $x$). The first letter can be anything we want it to be, often you will see $g(x)$ (read $g$ of $x$).

Once we know a relationship is a function, we may be interested in what values can be put into the equations. The values that are put into an equation (generally the $x$ values) are called the **domain**. When finding the domain, often it is easier to consider what can't happen in a given function, then exclude those values.

**Example 4.**

$$\begin{array}{rl} \text{Find the domain: } f(x) = \dfrac{3x-1}{x^2+x-6} & \text{With fractions, zero can}'t \text{ be in denominator} \\ x^2 + x - 6 \neq 0 & \text{Solve by factoring} \\ (x+3)(x-2) \neq 0 & \text{Set each factor not equal to zero} \end{array}$$

$$\begin{array}{rl}
x+3\neq 0 \text{ and } x-2\neq 0 & \text{Solve each equation} \\
\underline{-3-3} \qquad\qquad \underline{+2+2} & \\
x\neq -3,2 & \text{Our Solution}
\end{array}$$

The notation in the previous example tells us that $x$ can be any value except for $-3$ and 2. If $x$ were one of those two values, the function would be undefined.

**Example 5.**

$$\begin{array}{rl}
\text{Find the domain: } f(x)=3x^2-x & \text{With this equation there are no bad values} \\
\text{All Real Numbers or } \mathbb{R} & \text{Our Solution}
\end{array}$$

In the above example there are no real numbers that make the function undefined. This means any number can be used for $x$.

**Example 6.**

$$\begin{array}{rl}
\text{Find the domain: } f(x)=\sqrt{2x-3} & \text{Square roots can't be negative} \\
2x-3\geqslant 0 & \text{Set up an inequality} \\
+3+3 & \text{Solve} \\
2x\geqslant 3 & \\
\overline{2}\ \ \overline{2} & \\
x\geqslant \frac{3}{2} & \text{Our Solution}
\end{array}$$

The notation in the above example states that our variable can be $\frac{3}{2}$ or any number larger than $\frac{3}{2}$. But any number smaller would make the function undefined (without use imaginary numbers).

Another use of function notation is to easily plug values into functions. If we want to substitute a variable for a value (or an expression) we simplify replace the varible with what we want to plug in. This is shown in the following examples.

**Example 7.**

$$\begin{array}{rl}
f(x)=3x^2-4x;\text{find } f(-2) & \text{Substitute } -2 \text{ in for } x \text{ in the function} \\
f(-2)=3(-2)^2-4(-2) & \text{Evaluate, exponents first} \\
f(-2)=3(4)-4(-2) & \text{Multiply} \\
f(-2)=12+8 & \text{Add} \\
f(-2)=20 & \text{Our Solution}
\end{array}$$

**Example 8.**

$$\begin{aligned} h(x) = 3^{2x-6}; \text{find } h(4) \quad & \text{Substitute 4 in for } x \text{ in the function} \\ h(4) = 3^{2(4)-6} \quad & \text{Simplify exponent, mutiplying first} \\ h(4) = 3^{8-6} \quad & \text{Subtract in exponent} \\ h(4) = 3^2 \quad & \text{Evaluate exponent} \\ h(4) = 9 \quad & \text{Our Solution} \end{aligned}$$

**Example 9.**

$$\begin{aligned} k(a) = 2|a+4|; \text{find } k(-7) \quad & \text{Substitute } -7 \text{ in for } a \text{ in the function} \\ k(-7) = 2|-7+4| \quad & \text{Add inside absolute values} \\ k(-7) = 2|-3| \quad & \text{Evaluate absolute value} \\ k(-7) = 2(3) \quad & \text{Multiply} \\ k(-7) = 6 \quad & \text{Our Solution} \end{aligned}$$

As the above example show, the function can take many different forms, but the pattern to evaluate the function is always the same, replace the variable with what is in parenthesis and simplify. We can also substitute expressions into functions using the same process. Often the expressions use the same variable, it is important to remember each variable is replaced by whatever is in parenthesis.

**Example 10.**

$$\begin{aligned} g\,(x) = x^4 + 1; \text{find } f(3x) \quad & \text{Replace } x \text{ in the function with } (3x) \\ g(3x) = (3x)^4 + 1 \quad & \text{Simplify exponet} \\ g(3x) = 81x^4 + 1 \quad & \text{Our Solution} \end{aligned}$$

**Example 11.**

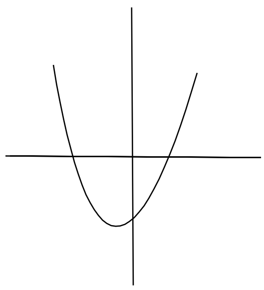
$$\begin{aligned} p(t) = t^2 - t; \text{find } p(t+1) \quad & \text{Replace each } t \text{ in } p(t) \text{ with } (t+1) \\ p(t+1) = (t+1)^2 - (t+1) \quad & \text{Square binomial} \\ p(t+1) = t^2 + 2t + 1 - (t+1) \quad & \text{Distribute negative} \\ p(t+1) = t^2 + 2t + 1 - t - 1 \quad & \text{Combine like terms} \\ p(t+1) = t^2 + t \quad & \text{Our Solution} \end{aligned}$$

It is important to become comfortable with function notation and how to use it as we transition into more advanced algebra topics.

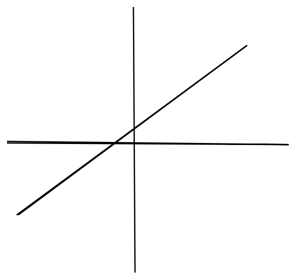
# Practice - Function Notation

1) Which of the following is a function?

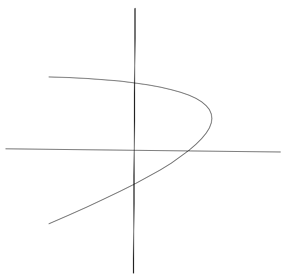
a)



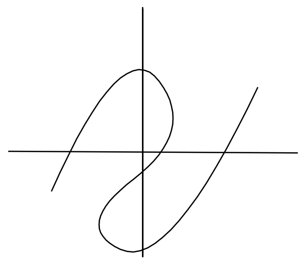
b)



c)



d)



e) $y = 3x - 7$

f) $y^2 - x^2 = 1$

g) $\sqrt{y} + x = 2$

h) $x^2 + y^2 = 1$

**Specify the domain of each of the following funcitons.**

2) $f(x) = -5x + 1$

3) $f(x) = \sqrt{5 - 4x}$

4) $s(t) = \frac{1}{t^2}$

5) $f(x) = x^2 - 3x - 4$

6) $s(t) = \frac{1}{t^2 + 1}$

7) $f(x) = \sqrt{x - 16}$

8) $f(x) = \frac{-2}{x^2 - 3x - 4}$

9) $h(x) = \frac{\sqrt{3x - 12}}{x^2 - 25}$

10 $y(x) = \frac{x}{x^2 - 25}$

**Evaluate each function.**

11) $g(x) = 4x - 4; \text{Find } g(0)$

13) $f(x) = |3x+1| + 1$; Find $f(0)$

14) $f(x) = x^2 + 4$; Find $f(-9)$

15) $f(n) = -2\big|-n-2\big| + 1$; Find $f(-6)$

16) $f(n) = n - 3$; Find $f(10)$

17) $f(t) = 3^t - 2$; Find $f(-2)$

18) $f(a) - 3^{a-1} - 3$; Find $f(2)$

19) $f(t) = |t+3|$; Find $f(10)$

20) $w(x) = x^2 + 4x$; Find $w(-5)$

21) $w(n) = 4n + 3$; Find $w(2)$

22) $w(x) = -4x + 3$; Find $w(6)$

23) $w(n) = 2^{n+2}$; Find $w(-2)$

24) $p(x) = -|x| + 1$; Find $p(5)$

25) $p(n) = -3|n|$; Find $p(7)$

26) $k(a) = a + 3$; Find $k(-1)$

27) $p(t) = -t^3 + t$; Find $p(4)$

28) $k(x) = -2 \cdot 4^{2x-2}$; Find $k(2)$

29) $k(n) = |n-1|$; Find $k(3)$

30) $p(t) = -2 \cdot 4^{2t+1} + 1$; Find $p(-2)$

31) $h(x) = x^3 + 2$; Find $h(-4x)$

32) $h(n) = 4n + 2$; Find $h(n+2)$

33) $h(x) = 3x + 2$; Find $h(-1+x)$

34) $h(a) = -3 \cdot 2^{a+3}$; Find $h(\frac{a}{4})$

35) $h(t) = 2\big|-3t-1\big| + 2$; Find $h(n^2)$

36) $h(x) = x^2 + 1$; Find $h(\frac{x}{4})$

37) $g(x) = x + 1$; Find $g(3x)$

38) $h(t) = t^2 + t$; Find $h(t^2)$

39) $g(x) = 5^x$; Find $g(-3-x)$

12) $g(n) = -3 \cdot 5^{-n}$; Find $g(2)$

40) $h(n) = 5^{n-1} + 1$; Find $h(\frac{n}{2})$

10.2

# Functions - Algebra of Functions

Several functions can work together in one larger function. There are 5 common operations that can be performed on functions. The four basic operations on functions are adding, subtracting, multiplying, and dividing. The notation for these functions is as follows.

$$\begin{array}{rl} \text{Addition} & (f+g)(x)=f(x)+g(x) \\ \text{Subtraction} & (f-g)(x)=f(x)-g(x) \\ \text{Multiplication} & (f\cdot g)(x)=f(x)g(x) \\ \text{Division} & \left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)} \end{array}$$

When we do one of these four basic operations we can simply evaluate the two functions at the value and then do the operation with both solutions

**Example 12.**

$$\begin{array}{ll} f(x)=x^2-x-2 & \\ g(x)=x+1 & \text{Evaluate } f \text{ and } g \text{ at } -3 \\ \text{find } (f+g)(-3) & \end{array}$$

$$\begin{array}{rl} f(-3)=(-3)^2-(-3)-3 & \text{Evaluate } f \text{ at } -3 \\ f(-3)=9+3-3 & \\ f(-3)=9 & \end{array}$$

$$\begin{array}{rl} g(-3)=(-3)+1 & \text{Evaluate } g \text{ at } -3 \\ g(-3)=-2 & \end{array}$$

$$\begin{array}{rl} f(-3)+g(-3) & \text{Add the two functions together} \\ (9)+(-2) & \text{Add} \\ 7 & \text{Our Solution} \end{array}$$

The process is the same regardless of the operation being performed.

**Example 13.**

$$\begin{array}{ll} h(x)=2x-4 & \\ k(x)=-3x+1 & \text{Evaluate } h \text{ and } k \text{ at } 5 \\ \text{Find } (h\cdot k)(5) & \end{array}$$

$$\begin{array}{rl} h(5)=2(5)-4 & \text{Evaluate } h \text{ at } 5 \\ h(5)=10-4 & \end{array}$$

$$h(5) = 6$$

$$\begin{aligned} k(5) &= -3(5) + 1 && \text{Evaluate } k \text{ at } 5 \\ k(5) &= -15 + 1 \\ k(5) &= -14 \end{aligned}$$

$$\begin{aligned} h(5)k(5) &\quad \text{Multiply the two results together} \\ (6)(-14) &\quad \text{Multiply} \\ -84 &\quad \text{Our Solution} \end{aligned}$$

Often as we add, subtract, multiply, or divide functions, we do so in a way that keeps the variable. If there is no number to plug into the equations we will simply use each equation, in parenthesis, and simplify the expression.

**Example 14.**

$$\begin{aligned} f(x) &= 2x - 4 \\ g(x) &= x^2 - x + 5 && \text{Write subtraction problem of functions} \\ \text{Find } &(f - g)(x) \end{aligned}$$

$$\begin{aligned} f(x) - g(x) &\quad \text{Replace } f(x) \text{ with } (2x - 3) \text{ and } g(x) \text{ with } (x^2 - x + 5) \\ (2x - 3) - (x^2 - x + 5) &\quad \text{Distribute the negative} \\ 2x - 3 - x^2 + x - 5 &\quad \text{Combine like terms} \\ -x^2 + 3x - 8 &\quad \text{Our Solution} \end{aligned}$$

The parenthesis aer very important when we are replacing $f(x)$ and $g(x)$ with a variable. In the previous example we needed the parenthesis to know to distribute the negative.

**Example 15.**

$$\begin{aligned} &f(x) = x^2 - 4x - 5 \\ &g(x) = x - 5 \qquad\qquad \text{Write division problem of functions} \\ &\text{Find } \left(\frac{f}{g}\right)(x) \end{aligned}$$

$$\frac{f(x)}{g(x)} \quad \text{Replace } f(x) \text{ with } (x^2 - 4x - 5) \text{ and } g(x) \text{ with } (x - 5)$$

$$\frac{(x^2 - 4x - 5)}{(x - 5)} \quad \text{To simplify the fraction we must first factor}$$

$$\frac{(x - 5)(x + 1)}{(x - 5)} \quad \text{Divide out common factor of } x - 5$$

$$x+1 \quad \text{Our Solution}$$

Just as we could substitute an expression into evaluating functions, we can substitute an expression into the operations on functions.

**Example 16.**

$$\begin{aligned} f(x) &= 2x-1 \\ g(x) &= x+4 && \text{Write as } a \text{ sum of functions} \\ \text{Find } (f+g)(x^2) \end{aligned}$$

$$\begin{aligned} f(x^2)+g(x^2) & \quad \text{Replace } x \text{ in } f(x) \text{ and } g(x) \text{ with } x^2 \\ [2(x^2)-1]+[(x^2)+4] & \quad \text{Distribute the } + \text{ does not change the problem} \\ 2x^2-1+x^2+4 & \quad \text{Combine like terms} \\ 3x^2+3 & \quad \text{Our Solution} \end{aligned}$$

**Example 17.**

$$\begin{aligned} f(x) &= 2x-1 \\ g(x) &= x+4 && \text{Write as } a \text{ product of functions} \\ \text{Find } (f\cdot g)(3x) \end{aligned}$$

$$\begin{aligned} f(3x)g(3x) & \quad \text{Replace } x \text{ in } f(x) \text{ and } g(x) \text{ with } 3x \\ [2(3x)-1][(3x)+4] & \quad \text{Multiply our } 2(3x) \\ (6x-1)(3x+4) & \quad \text{FOIL} \\ 18x^2+24x-3x-4 & \quad \text{Combine like terms} \\ 18x^2+21x-4 & \quad \text{Our Solution} \end{aligned}$$

The fifth operation of functions is called composition of functions. A composition of functions is a function inside of a function. The notation used for composition of functions is:

$$\boldsymbol{(f\circ g)(x)=f(g(x))}$$

To calculate a composition of function we will evaluate the inner function and substitute the answer into the outer function. This is shown in the following example.

**Example 18.**

$$\begin{aligned} a(x) &= x^2-2x+1 \\ b(x) &= x-5 && \text{Rewrite as } a \text{ function in function} \\ \text{Find } (a\circ b)(3) \end{aligned}$$

$$
\begin{array}{rl}
a(b(3)) & \text{Evaluate the inner function first, } b(3) \\
b(3) = (3) - 5 = -2 & \text{This solution is put into } a, a(-2) \\
a(-2) = (-2)^2 - 2(-2) + 1 & \text{Evaluate} \\
a(-2) = 4 + 4 + 1 & \text{Add} \\
a(-2) = 9 & \text{Our Solution}
\end{array}
$$

We can also evaluate a composition of functions at a variable. In these problems we will take the inside function and substitute into the outside function.

**Example 19.**

$$
\begin{array}{ll}
f(x) = x^2 - x & \\
g(x) = x + 3 & \text{Rewrite as } a \text{ function in function} \\
\text{Find } (f \circ g)(x) &
\end{array}
$$

$$
\begin{array}{rl}
f(g(x)) & \text{Replace } g(x) \text{ with } x + 3 \\
f(x + 3) & \text{Replace the variables in } f \text{ with } (x + 3) \\
(x + 3)^2 - (x + 3) & \text{Evaluate exponent} \\
(x^2 + 6x + 9) - (x + 3) & \text{Distirbute negative} \\
x^2 + 6x + 9 - x - 3 & \text{Combine like terms} \\
x^2 + 5x + 6 & \text{Our Solution}
\end{array}
$$

It is important to note that very rarely is $(f \circ g)(x)$ the same as $(g \circ f)(x)$ as the following example will show, using the same equations, but compositing them in the opposite direction.

**Example 20.**

$$
\begin{array}{ll}
f(x) = x^2 - x & \\
g(x) = x + 3 & \text{Rewrite as } a \text{ function in function} \\
\text{Find } (g \circ f)(x) &
\end{array}
$$

$$
\begin{array}{rl}
g(f(x)) & \text{Replace } f(x) \text{ with } x^2 - x \\
g(x^2 - x) & \text{Replace the variable in } g \text{ with } (x^2 - x) \\
(x^2 - x) + 3 & \text{Here the parenthesis don}'t \text{ change the expression} \\
x^2 - x + 3 & \text{Our Solution}
\end{array}
$$

# Practice - Algebra of Functions

**Perform the indicated operations.**

1) $g(a)=a^3+5a^2$
$f(a)=2a+4$
Find $g(3)+f(3)$

3) $g(a)=3a+3$
$f(a)=2a-2$
Find $(g+f)(9)$

5) $g(x)=x+3$
$f(x)=-x+4$
Find $(g-f)(3)$

7) $g(x)=x^2+2$
$f(x)=2x+5$
Find $(g-f)(0)$

9) $g(t)=t-3$
$h(t)=-3t^3+6t$
Find $g(1)+h(1)$

11) $h(t)=t+5$
$g(t)=3t-5$
Find $(h\cdot g)(5)$

13) $h(n)=2n-1$
$g(n)=3n-5$
Find $h(0)\div g(0)$

15) $f(a)=-2a-4$
$g(a)=a^2+3$
Find $(\frac{f}{g})(7)$

17) $g(x)=-x^3-2$
$h(x)=4x$
Find $(g-h)(x)$

19) $f(x)=-3x+2$
$g(x)=x^2+5x$
Find $(f-g)(x)$

21) $g(x)=4x+5$
$h(x)=x^2+5x$
Find $g(x)\cdot h(x)$

23) $f(x)=x^2-5x$
$g(x)=x+5$
Find $(f+g)(x)$

25) $g(n)=n^2+5$
$f(n)=3n+5$
Find $g(n)\div f(n)$

27) $g(a)=-2a+5$
$f(a)=3a+5$
Find $(\frac{g}{f})(a)$

29) $h(n)=n^3+4n$
$g(n)=4n+5$
Find $h(n)+g(n)$

31) $g(n)=n^2-4n$
$h(n)=n-5$
Find $g(n^2)\cdot h(n^2)$

33) $f(x)=2x$
$g(x)=-3x-1$
Find $(f+g)(-4-x)$

35) $f(t)=t^2+4t$
$g(t)=4t+2$
Find $f(t^2)+g(t^2)$

37) $g(a)=a^3+2a$
$h(a)=3a+4$
Find $(\frac{g}{h})(-x)$

39) $f(n)=-3n^2+1$
$g(n)=2n+1$
Find $(f-g)(\frac{n}{3})$

41) $f(x)=-4x+1$
$g(x)=4x+3$
Find $(f\circ g)(9)$

43) $h(a)=3a+3$
$g(a)=a+1$

Find $(h \circ g)(5)$

45) $g(x) = x + 4$
$h(x) = x^2 - 1$
Find $(g \circ h)(10)$

47) $f(n) = -4n + 2$
$g(n) = n + 4$
Find $(f \circ g)(9)$

49) $g(x) = 2x - 4$
$h(x) = 2x^3 + 4x^2$
Find $(g \circ h)(3)$

51) $g(x) = x^2 - 5x$
$h(x) = 4x + 4$
Find $(g \circ h)(x)$

53) $f(a) = -2a + 2$
$g(a) = 4a$
Find $(f \circ g)(a)$

55) $g(x) = 4x + 4$
$f(x) = x^3 - 1$
Find $(g \circ f)(x)$

57) $g(x) = -x + 5$
$f(x) = 2x - 3$
Find $(g \circ f)(x)$

59) $f(t) = 4t + 3$
$g(t) = -4t - 2$
Find $(f \circ g)(t)$

2) $f(x) = -3x^2 + 3x$
$g(x) = 2x + 5$
Find $f(-4) \div g(-4)$

4) $g(x) = 4x + 3$
$h(x) = x^3 - 2x^2$
Find $(g - h)(-1)$

6) $g(x) = -4x + 1$
$h(x) = -2x - 1$
Find $g(5) + h(5)$

8) $g(x) = 3x + 1$
$f(x) = x^3 + 3x^2$
Find $g(2) \cdot f(2)$

10) $f(n) = n - 5$
$g(n) = 4n + 2$
Find $(f + g)(-8)$

12) $g(a) = 3a - 2$
$h(a) = 4a - 2$
Find $(g + h)(-10)$

14) $g(x) = x^2 - 2$
$h(x) = 2x + 5$
Find $g(-6) + h(-6)$

16) $g(n) = n^2 - 3$
$h(n) = 2n - 3$
Find $(g - h)(n)$

18) $g(x) = 2x - 3$
$h(x) = x^3 - 2x^2 + 2x$
Find $(g - h)(x)$

20) $g(t) = t - 4$
$h(t) = 2t$
Find $(g \cdot h)(t)$

22) $g(t) = -2t^2 - 5t$
$h(t) = t + 5$
Find $g(t) \cdot h(t)$

24) $f(x) = 4x - 4$
$g(x) = 3x^2 - 5$
Find $(f + g)(x)$

26) $f(x) = 2x + 4$
$g(x) = 4x - 5$
Find $f(x) - g(x)$

28) $g(t) = t^3 + 3t^2$
$h(t) = 3t - 5$
Find $g(t) - h(t)$

30) $f(x) = 4x + 2$
$g(x) = x^2 + 2x$
Find $f(x) \div g(x)$

32) $g(n) = n + 5$
$h(n) = 2n - 5$
Find $(g \cdot h)(-3n)$

34) $g(a) = -2a$
$h(a) = 3a$

Find $g(4n) \div h(4n)$

36) $h(n) = 3n - 2$
$g(n) = -3n^2 - 4n$
Find $h(\frac{n}{3}) \div g(\frac{n}{3})$

38) $g(x) = -4x + 2$
$h(x) = x^2 - 5$
Find $g(x^2) + h(x^2)$

40) $f(n) = 3n + 4$
$g(n) = n^3 - 5n$
Find $f(\frac{n}{2}) - g(\frac{n}{2})$

42) $g(x) = x - 1$
Find $(g \circ g)(7)$

44) $g(t) = t + 3$
$h(t) = 2t - 5$
Find $(g \circ h)(3)$

46) $f(a) = 2a - 4$
$g(a) = a^2 + 2a$
Find $(f \circ g)(-4)$

48) $g(x) = 3x + 4$
$h(x) = x^3 + 3x$
Find $(g \circ h)(3)$

50) $g(a) = a^2 + 3$
Find $(g \circ g)(-3)$

52) $g(a) = 2a + 4$
$h(a) = -4a + 5$
Find $(g \circ h)(a)$

54) $g(t) = -t - 4$
Find $(g \circ g)(t)$

56) $f(n) = -2n^2 - 4n$
$g(n) = n + 2$
Find $(f \circ g)(n)$

58) $g(t) = t^3 - t$
$f(t) = 3t - 4$
Find $(g \circ f)(t)$

60) $f(x) = 3x - 4$
$g(x) = x^3 + 2x^2$
Find $(f \circ g)(x)$

10.3

# Functions - Inverse

When a value goes into a function it is called the input. The result that we get when we evaluate the function is called the output. When working with functions sometimes we will know the output and be interested in what input gave us the output. To find this we use an inverse function. As the name suggests an inverse function undoes whatever the function did. If a function is named $f(x)$, the inverse function will be named $f^{-1}(x)$ (read "$f$ inverse of $x$"). The negative one is not an exponent, but mearly a symbol to let us know that this function is the inverse of $f$.

For example, if $f(x) = x + 5$, we could deduce that the inverse function would be $f^{-1}(x) = x - 5$. If we had an input of 3, we could calculate $f(3) = (3) + 5 = 8$. Our output is 8. If we plug this output into the inverse function we get $f^{-1}(8) = (8) - 5 = 3$, which is the original input.

Often the functions are much more involved than those described above. It may be difficult to determine just by looking at the functions if they are inverses. In order to test if two functions, $f(x)$ and $g(x)$ are inverses we will calculate the composition of the two functions at $x$. If $f$ changes the variable $x$ in some way, then $g$ undoes whatever $f$ did, then we will be back at $x$ again for our final solution. In otherwords if we simplify $(f \circ g)(x)$ the solution will be $x$. If it is anything but $x$ the functions are not inverses.

**Example 21.**

$$
\begin{array}{rl}
\text{Are } f(x)=\sqrt[3]{3x+4} \text{ and } g(x)=\dfrac{x^3-4}{3} \text{ inverses?} & \text{Caculate composition} \\
f(g(x)) & \text{Replace } g(x) \text{ with } \dfrac{x^3-4}{3} \\
f\left(\dfrac{x^3-4}{3}\right) & \text{Substitute } \left(\dfrac{x^3-4}{3}\right) \text{ for variable in } f \\
\sqrt[3]{3\left(\dfrac{x^3-4}{3}\right)+4} & \text{Divide out the } 3's \\
\sqrt[3]{x^3-4+4} & \text{Combine like terms} \\
\sqrt[3]{x^3} & \text{Take cubed root} \\
x & \text{Simplified to } x! \\
\text{Yes, they are inverses!} & \text{Our Solution}
\end{array}
$$

**Example 22.**

$$
\begin{array}{rl}
\text{Are } h(x)=2x+5 \text{ and } g(x)=\dfrac{x}{2}-5 \text{ inverses?} & \text{Calculate composition} \\
h(g(x)) & \text{Replace } g(x) \text{ with } \left(\dfrac{x}{2}-5\right) \\
h\left(\dfrac{x}{2}-5\right) & \text{Substitute } \left(\dfrac{x}{2}-5\right) \text{ for variable in } h \\
2\left(\dfrac{x}{2}-5\right)+5 & \text{Distrubte } 2 \\
x-10+5 & \text{Combine like terms} \\
x-5 & \text{Did not simplify to } x \\
\text{No, they are not inverses} & \text{Our Solution}
\end{array}
$$

**Example 23.**

$$
\begin{array}{rl}
\text{Are } f(x)=\dfrac{3x-2}{4x+1} \text{ and } g(x)=\dfrac{x+2}{3-4x} \text{ inverses?} & \text{Calculate composition} \\
f(g(x)) & \text{Replace } g(x) \text{ with } \left(\dfrac{x+2}{3-4x}\right) \\
f\left(\dfrac{x+2}{3-4x}\right) & \text{Substitute } \left(\dfrac{x+2}{3-4x}\right) \text{ for variable in } f \\
\dfrac{3\left(\frac{x+2}{3-4x}\right)-2}{4\left(\frac{x+2}{3-4x}\right)+1} & \text{Distribute 3 and 4 into numerators}
\end{array}
$$

$$\frac{\frac{3x+6}{3-4x}-2}{\frac{4x+8}{3-4x}+1} \quad \text{Multiply each term by LCD: } 3-4x$$

$$\frac{\frac{(3x+6)(3-4x)}{3-4x}-2(3-4x)}{\frac{(4x+8)(3-4x)}{3-4x}+1(3-4x)} \quad \text{Reduce fractions}$$

$$\frac{3x+6-2(3-4x)}{4x+8+1(3-4x)} \quad \text{Distribute}$$

$$\frac{3x+6-6+8x}{4x+8+3-4x} \quad \text{Combine like terms}$$

$$\frac{11x}{11} \quad \text{Divide out 11}$$

$$x \quad \text{Simplified to } x!$$

Yes, they are inverses — Our Solution

While the composition is useful to show two functions are invereses, a more common problem is to find the inverse of a function. If we think of $x$ as our input and $y$ as our output from a function, then the inverse will take $y$ as an input and give $x$ as the output. This means if we switch $x$ and $y$ in our function we will find the inverse! This process is called the switch and solve strategy.

**Switch and solve strategy to find an inverse:**

1. Replace $f(x)$ with $y$
2. Switch $x$ and $y$'s
3. Solve for $y$
4. Replace $y$ with $f^{-1}(x)$

**Example 24.**

$$\begin{array}{rl}
\text{Find the inverse of } f(x)=(x+4)^3-2 & \text{Replace } f(x) \text{ with } y \\
y=(x+4)^3-2 & \text{Switch } x \text{ and } y \\
x=(y+4)^3-2 & \text{Solve for } y \\
\underline{+2 \qquad\qquad +2} & \text{Add 2 to both sides}
\end{array}$$

$$
\begin{array}{rl}
x+2=(y+4)^3 & \text{Cube root both sides} \\
\sqrt[3]{x+2}=y+4 & \text{Subtract 4 from both sides} \\
\underline{-4 \qquad -4} & \\
\sqrt[3]{x+2}-4=y & \text{Replace } y \text{ with } f^{-1}(x) \\
f^{-1}(x)=\sqrt[3]{x+2}-4 & \text{Our Solution}
\end{array}
$$

**Example 25.**

$$
\begin{array}{rl}
\text{Find the inverse of } g(x)=\dfrac{2x-3}{4x+2} & \text{Replace } g(x) \text{ with } y \\
y=\dfrac{2x-3}{4x+2} & \text{Switch } x \text{ and } y \\
x=\dfrac{2y-3}{4y+2} & \text{Multiply by } (4y+2) \\
x(4y+2)=2y-3 & \text{Distribute} \\
4xy+2x=2y-3 & \text{Move all } y's \text{ to one side, rest to other side} \\
\underline{-4xy+3-4xy+3} & \text{Subtract } 4xy \text{ and add 3 to both sides} \\
2x+3=2y-4xy & \text{Factor out } y \\
\dfrac{2x+3}{2-4x}=\dfrac{y(2-4x)}{2-4x} & \text{Divide by } 2-4x \\
\dfrac{2x+3}{2-4x}=y & \text{Replace } y \text{ with } g^{-1}(x) \\
g^{-1}(x)=\dfrac{2x+3}{2-4x} & \text{Our Solution}
\end{array}
$$

In this lesson we looked at two different things, first showing functions are inverses by calculating the composition, and second finding an inverse when we only have one function. Be careful not to get them backwards. When we already have two functions and are asked to show they are inverses, we do not want to use the switch and solve strategy, what we want to do is calculate the inverse. There may be several ways to represent the same function so the switch and solve startegy may not look the way we expect and can lead us to conclude two functions are not inverses when they are in fact inverses.

# Practice - Inverse Functions

**State if the given functions are inverses.**

1) $g(x)=-x^5-3$
$f(x)=\sqrt[5]{-x-3}$

2) $g(x)=\frac{4-x}{x}$
$f(x)=\frac{4}{x}$

3) $f(x)=\frac{-x-1}{x-2}$
$g(x)=\frac{-2x+1}{-x-1}$

4) $h(x)=\frac{-2-2x}{x}$
$f(x)=\frac{4}{x}$

5) $g(x)=-10x+5$
$f(x)=\frac{x-5}{10}$

6) $f(x)=\frac{x-5}{10}$
$h(x)=10x+5$

7) $f(x)=-\frac{2}{x+3}$
$g(x)=\frac{3x+2}{x+2}$

8) $f(x)=\sqrt[5]{\frac{x+1}{2}}$
$g(x)=2x^5-1$

9) $g(x)=\sqrt[5]{\frac{x-1}{2}}$
$f(x)=2x^5+1$

10) $g(x)=\frac{8+9x}{2}$
$f(x)=\frac{5x-9}{2}$

**Find the inverse of each functions.**

11) $f(x)=(x-2)^5+3$

12) $g(x)=\sqrt[3]{x+1}+2$

13) $g(x)=\frac{4}{x+2}$

14) $f(x)=\frac{-3}{x-3}$

15) $f(x)=\frac{-2x-2}{x+2}$

16) $g(x)=\frac{9+x}{3}$

17) $f(x)=\frac{10-x}{5}$

18) $f(x)=\frac{5x-15}{2}$

19) $g(x)=-(x-1)^3$

20) $f(x)=\frac{12-3x}{4}$

21) $f(x)=(x-3)^3$

22) $g(x)=\sqrt[5]{\frac{-x+2}{2}}$

23) $g(x)=\frac{x}{x-1}$

24) $f(x)=\frac{-3-2x}{x+3}$

25) $f(x)=\frac{x-1}{x+1}$

26) $h(x)=\frac{x}{x+2}$

27) $g(x)=\frac{8-5x}{4}$

28) $g(x)=\frac{-x+2}{3}$

29) $g(x)=-5x+1$

30) $f(x)=\frac{5x-5}{4}$

31) $g(x)=-1+x^3$

32) $f(x)=3-2x^5$

33) $h(x)=\frac{4-\sqrt[3]{4x}}{2}$

34) $g(x)=(x-1)^3+2$

35) $f(x)=\frac{x+1}{x+2}$

36) $f(x)=\frac{-1}{x+1}$

37) $f(x)=\frac{7-3x}{x-2}$

38) $f(x)=-\frac{3x}{4}$

39) $g(x)=-x$

40) $g(x)=\frac{-2x+1}{3}$

10.4

# Functions - Exponential Functions

As our study of algebra gets more advanced we begin to study more involved functions. One pair of inverse functions we will look at are exponential functions and logarithmic functions. Here we will look at exponential functions and then we will consider logarithmic functions in another lesson. Exponential functions are functions where the variable is in the exponent such as $f(x) = a^x$. (It is important not to confuse exponential functions with polynomial functions where the variable is in the base such as $f(x) = x^2$).

Solving exponetial equations cannot be done using the skill set we have seen in the past. For example, if $3^x = 9$, we can't take the $x-$root of 9 because we don't know what the index is and this doesn't get us any closer to finding $x$. However, we may notice that 9 is $3^2$. We can then conclude that if $3^x = 3^2$ then $x = 2$. This is the process we will use to solve exponential functions. If we can re-write a problem so the bases match, then the exponents must also match.

**Example 26.**

$$\begin{aligned}
5^{2x+1} &= 125 && \text{Rewrite } 125 \text{ as } 5^3\\
5^{2x+1} &= 5^3 && \text{Same base, set exponents equal}\\
2x+1 &= 3 && \text{Solve}\\
\underline{-1} &\,\underline{-1} && \text{Subtract 1 from both sdies}\\
2x &= 2 && \text{Divide both sides by 2}\\
\overline{\ 2\ } &\ \overline{\ 2\ } && \\
x &= 1 && \text{Our Solution}
\end{aligned}$$

Sometimes we may have to do work on both sides of the equation to get a common base. As we do so, we will use various exponent properties to help. First we will use the exponent property that that states $(a^x)^y = a^{xy}$.

**Example 27.**

$$\begin{aligned}
8^{3x} &= 32 && \text{Rewrite 8 as } 2^3 \text{ and 32 as } 2^5\\
(2^3)^{3x} &= 2^5 && \text{Multiply exponents 3 and } 3x\\
2^{9x} &= 2^5 && \text{Same base, set exponents equal}\\
9x &= 5 && \text{Solve}\\
\overline{\ 9\ } &\ \overline{\ 9\ } && \text{Divide both sides by 9}\\
x &= \frac{5}{9} && \text{Our Solution}
\end{aligned}$$

As we multiply exponents we may need to distribute if there are several terms involved.

**Example 28.**

$$
\begin{array}{rl}
27^{3x+5} = 81^{4x+1} & \text{Rewrite 27 as } 3^3 \text{ and 81 as } 3^4 \text{ (} 9^2 \text{ would not be same base)} \\
(3^3)^{3x+5} = (3^4)^{4x+1} & \text{Multiply exponents } 3(3x+5) \text{ and } 4(4x+1) \\
3^{9x+15} = 3^{16x+4} & \text{Same base, set exponents equal} \\
9x + 15 = 16x + 4 & \text{Move variables to one side} \\
\underline{-9x \qquad\quad -9x\;\;} & \text{Subtract } 9x \text{ from both sides} \\
15 = 7x + 4 & \text{Subtract 4 from both sides} \\
\underline{-4 \qquad -4} & \\
11 = 7x & \text{Divide both sides by 7} \\
\overline{7} \quad \overline{7} & \\
\dfrac{11}{7} = x & \text{Our Solution}
\end{array}
$$

Another useful exponent property is that negative exponents will give us a reciprocal, $\frac{1}{a^n} = a^{-n}$

**Example 29.**

$$
\begin{array}{rl}
\left(\dfrac{1}{9}\right)^{2x} = 3^{7x-1} & \text{Rewrite } \dfrac{1}{9} \text{ as } 3^{-2} \text{ (negative exponet to flip)} \\
(3^{-2})^{2x} = 3^{7x-1} & \text{Multiply exponents } -2 \text{ and } 2x \\
3^{-4x} = 3^{7x-1} & \text{Same base, set exponets equal} \\
-4x = 7x - 1 & \text{Subtract } 7x \text{ from both sides} \\
\underline{-7x - 7x} & \\
-11x = -1 & \text{Divide by } -11 \\
\overline{-11} \quad \overline{-11} & \\
x = \dfrac{1}{11} & \text{Our Solution}
\end{array}
$$

If we have several factors with the same base on one side of the equation we can add the exponents using the property that states $a^x a^y = a^{x+y}$.

**Example 30.**

$$
\begin{array}{rl}
5^{4x} \cdot 5^{2x-1} = 5^{3x+11} & \text{Add exponents on left, combing like terms} \\
5^{6x-1} = 5^{3x+11} & \text{Same base, set exponents equal} \\
6x - 1 = 3x + 11 & \text{Move variables to one sides}
\end{array}
$$

$$\begin{array}{rl}
\underline{-3x \qquad -3x} & \text{Subtract } 3x \text{ from both sides} \\
3x-1=11 & \text{Add 1 to both sides} \\
\underline{+1 \quad +1} & \\
3x=12 & \text{Divide both sides by 3} \\
\overline{3} \quad \overline{3} & \\
x=4 & \text{Our Solution}
\end{array}$$

It may take a bit of practice to get use to knowing which base to use, but as we practice we will get much quicker at knowing which base to use. As we do so, we will use our exponent properties to help us simplify. Again, below are the properties we used to simplify.

$$\boldsymbol{(a^x)^y = a^{xy} \quad \text{and} \quad \frac{1}{a^n} = a^{-n} \quad \text{and} \quad a^x a^y = a^{x+y}}$$

We could see all three properties used in the same problem as we get a common base. This is shown in the next example.

**Example 31.**

$$\begin{array}{rl}
16^{2x-5} \cdot \left(\frac{1}{4}\right)^{3x+1} = 32 \cdot \left(\frac{1}{2}\right)^{x+3} & \text{Write with } a \text{ common base of 2} \\
(2^4)^{2x-5} \cdot (2^{-2})^{3x+1} = 2^5 \cdot (2^{-1})^{x+3} & \text{Multiply exponents, distributing as needed} \\
2^{8x-20} \cdot 2^{-6x-2} = 2^5 \cdot 2^{-x-3} & \text{Add exponents, combing like terms} \\
2^{2x-22} = 2^{-x+2} & \text{Same base, set exponents equal} \\
2x-22 = -x+2 & \text{Move variables to one side} \\
\underline{+x \qquad\quad +x} & \text{Add } x \text{ to both sides} \\
3x-22=2 & \text{Add 22 to both sides} \\
\underline{+22+22} & \\
3x=24 & \text{Divide both sides by 3} \\
\overline{3} \quad \overline{3} & \\
x=8 & \text{Our Solution}
\end{array}$$

All the problems we have solved here we were able to write with a common base. However, not all problems can be written with a common base, for example, $2 = 10^x$, we cannot write this problem with a common base. To solve problems like this we will need to use the inverse of an exponential function. The inverse is called a logarithmic function, which we will discuss in another secion.

# Practice - Exponential Functions

**Solve each equation.**

1) $3^{1-2n}=3^{1-3n}$

2) $4^{2x}=\frac{1}{16}$

3) $4^{2a}=1$

4) $16^{-3p}=64^{-3p}$

5) $(\frac{1}{25})^{-k}=125^{-2k-2}$

6) $625^{-n-2}=\frac{1}{125}$

7) $6^{2m+1}=\frac{1}{36}$

8) $6^{2r-3}=6^{r-3}$

9) $6^{-3x}=36$

10) $5^{2n}=5^{-n}$

11) $64^{b}=2^{5}$

12) $216^{-3v}=36^{3v}$

13) $(\frac{1}{4})^{x}=16$

14) $27^{-2n-1}=9$

15) $4^{3a}=4^{3}$

16) $4^{-3v}=64$

17) $36^{3x}=216^{2x+1}$

18) $64^{x+2}=16$

19) $9^{2n+3}=243$

20) $16^{2k}=\frac{1}{64}$

21) $3^{3x-2}=3^{3x+1}$

22) $243^{p}=27^{-3p}$

23) $3^{-2x}=3^{3}$

24) $4^{2n}=4^{2-3n}$

25) $5^{m+2}=5^{-m}$

26) $625^{2x}=25$

27) $(\frac{1}{36})^{b-1}=216$

28) $216^{2n}=36$

29) $6^{2-2x}=6^{2}$

30) $(\frac{1}{4})^{3v-2}=64^{1-v}$

31) $4\cdot 2^{-3n-1}=\frac{1}{4}$

32) $\frac{216}{6^{-2a}}=6^{3a}$

33) $4^{3k-3}\cdot 4^{2-2k}=16^{-k}$

34) $32^{2p-2}\cdot 8^{p}=(\frac{1}{2})^{2p}$

35) $9^{-2x}\cdot(\frac{1}{243})^{3x}=243^{-x}$

36) $3^{2m}\cdot 3^{3m}=1$

37) $64^{n-2}\cdot 16^{n+2}=(\frac{1}{4})^{3n-1}$

38) $3^{2-x}\cdot 3^{3m}=1$

39) $5^{-3n-3}\cdot 5^{2n}=1$

40) $4^{3r}\cdot 4^{-3r}=\frac{1}{64}$

10.5

# Functions - Logarithms

The inverse of an exponential function is a new function known as a logarithm. Lograithms are studied in detail in advanced algebra, here we will take an introductory look at how logarithms works. When working with radicals we found that their were two ways to write radicals. The expression $\sqrt[m]{a^n}$ could be written as $a^{\frac{n}{m}}$. Each form at its advantages, thus we had need to be comfortable using both the radical form and the rational exponent form. Similarly an exponent can be written in two forms, each with its own advantages. The first form we are very familiar with, $b^x = a$, where $b$ is the base, $a$ can be though of as our answer, and $x$ is the exponet. The second way to write this is with a logarithm, $\log_b a = x$. The word "log" tells us that we are in this new form. The variables all still mean the same thing. $b$ is still the base, $a$ can still be thought of as our answer.

Using this idea the problem $5^2 = 25$ could also be written as $\log_5 25 = 2$. Both mean the same thing, both are still the same exponent problem, but just as roots can be written in radical form or rational exponent form, both our forms have their own advantages. The most important thing to be comfortable doing with logarithms and exponents is to be able to switch back and forth between the two forms. This is what is shown in the next few examples.

**Example 32.**

Write each exponential equation in logarithmic form

$$\begin{array}{rl} m^3 = 5 & \text{Identify base, } m\text{, answer, } 5\text{, and exponent } 3 \\ \log_m 5 = 3 & \text{Our Solution} \end{array}$$

$$\begin{array}{rl} 7^2 = b & \text{Identify base, } 7\text{, answer, } b\text{, and exponent, } 2 \\ \log_7 b = 2 & \text{Our Solution} \end{array}$$

$$\begin{array}{rl} \left(\frac{2}{3}\right)^4 = \frac{16}{81} & \text{Identify base, } \frac{2}{3}\text{, answer, } \frac{16}{81}\text{, and exponent } 4 \\ \log_{\frac{2}{3}} \frac{16}{81} = 4 & \text{Our Solution} \end{array}$$

**Example 33.**

Write each logarithmic equation in exponential form

$$\begin{array}{rl} \log_4 16 = 2 & \text{Identify base, } 4\text{, answer, } 16\text{, and exponent, } 2 \\ 4^2 = 16 & \text{Our Solution} \end{array}$$

$$\begin{array}{rl} \log_3 x = 7 & \text{Idenfity base, 3, answer, } x\text{, and exponent, 7} \\ 3^7 = x & \text{Our Solution} \end{array}$$

$$\begin{array}{rl} \log_9 3 = \dfrac{1}{2} & \text{Idenfity base, 9, answer, 3, and exponent, } \dfrac{1}{2} \\ 9^{\frac{1}{2}} = 3 & \text{Our Solution} \end{array}$$

Once we are comfortable switching between logarithmic and exponential form we are able to evaluate and solve logarithmic expressions and equations. We will first evaluate logarithmic expressions. An easy way to evaluate a logarithm is to set the logarithm equal to $x$ and change it into an exponential equation.

**Example 34.**

$$\begin{array}{rl} \text{Evaluate } \log_2 64 & \text{Set logarithm equal to } x \\ \log_2 64 = x & \text{Change to exponent form} \\ 2^x = 64 & \text{Write as common base, } 64 = 2^6 \\ 2^x = 2^6 & \text{Same base, set exponents equal} \\ x = 6 & \text{Our Solution} \end{array}$$

**Example 35.**

$$\begin{array}{rl} \text{Evaluate } \log_{125} 5 & \text{Set logarithm equal to } x \\ \log_{125} 5 = x & \text{Change to exponent form} \\ 125^x = 5 & \text{Write as common base, } 125 = 5^3 \\ (5^3)^x = 5 & \text{Multiply exponnets} \\ 5^{3x} = 5 & \text{Same base, set exponnets equal } (5 = 5^1) \\ 3x = 1 & \text{Solve} \\ \overline{3} \quad \overline{3} & \text{Divide both sides by 3} \\ x = \dfrac{1}{3} & \text{Our Solution} \end{array}$$

**Example 36.**

$$\begin{array}{rl} \text{Evaluate } \log_3 \dfrac{1}{27} & \text{Set logarithm equal to } x \\ \log_3 \dfrac{1}{27} = x & \text{Change to exponent form} \\ 3^x = \dfrac{1}{27} & \text{Write as common base, } \dfrac{1}{27} = 3^{-3} \\ 3^x = 3^{-3} & \text{Same base, set exponents equal} \\ x = -3 & \text{Our Solution} \end{array}$$

Solve equations with logarithms is done in a very similar way, we simply will change the equation into exponential form and try to solve the resulting equation.

**Example 37.**

$$
\begin{aligned}
\log_5 x &= 2 && \text{Change to exponential form}\\
5^2 &= x && \text{Evaluate exponent}\\
25 &= x && \text{Our Solution}
\end{aligned}
$$

**Example 38.**

$$
\begin{aligned}
\log_2(3x+5) &= 4 && \text{Change to exponential form}\\
2^4 &= 3x+5 && \text{Evaluate exponent}\\
16 &= 3x+5 && \text{Solve}\\
\underline{-5} &\quad \underline{-5} && \text{Subtract 5 from both sides}\\
11 &= 3x && \text{Divide both sides by 3}\\
\overline{3} &\quad \overline{3} &&\\
\frac{11}{3} &= x && \text{Our Solution}
\end{aligned}
$$

**Example 39.**

$$
\begin{aligned}
\log_x 8 &= 3 && \text{Change to exponential form}\\
x^3 &= 8 && \text{Cube root of both sides}\\
x &= 2 && \text{Our Solution}
\end{aligned}
$$

There is one base on a logarithm that gets used more often than any other base, base 10. Similar to square roots not writting the common index of 2 in the radical, we don't write the common base of 10 in the logarithm. So if we are working on a problem with no base written we will always assume that base is base 10.

**Example 40.**

$$
\begin{aligned}
\log x &= -2 && \text{Rewire as exponent, 10 is base}\\
10^{-2} &= x && \text{Evaluate, remember negative exponent is fraction}\\
\frac{1}{100} &= x && \text{Our Solution}
\end{aligned}
$$

This lesson has introduced the idea of logarithms, changing between logs and exponents, evaluating logarithms, and solving basic logarithmic equations. In an advanced algebra course logarithms will be studied in much greater detail.

# Practice - Logarithmic Functions

**Rewrite each equation in exponential form.**

1) $\log_9 81 = 2$

2) $\log_b a = -16$

3) $\log_7 \frac{1}{49} = -2$

4) $\log_{16} 256 = 2$

5) $\log_{13} 169 = 2$

6) $\log_{11} 1 = 0$

**Rewrite each equations in logarithmic form.**

7) $8^0 = 1$

8) $17^{-2} = \frac{1}{289}$

9) $15^2 = 225$

10) $144^{\frac{1}{2}} = 12$

11) $64^{\frac{1}{6}} = 2$

12) $19^2 = 361$

**Evaluate each expression.**

13) $\log_{125} 5$

14) $\log_5 125$

15) $\log_{343} \frac{1}{7}$

16) $\log_7 1$

17) $\log_4 16$

18) $\log_4 \frac{1}{64}$

19) $\log_6 36$

20) $\log_{36} 6$

21) $\log_2 64$

22) $\log_3 243$

**Solve each equation.**

23) $\log_5 x = 1$

24) $\log_8 k = 3$

25) $\log_2 x = -2$

26) $\log n = 3$

27) $\log_{11} k = 2$

28) $\log_4 p = 4$

29) $\log_9 (n + 9) = 4$

30) $\log_{11} (x - 4) = -1$

31) $\log_5 -3m = 3$

32) $\log_2 -8r = 1$

33) $\log_{11} (x + 5) = -1$

34) $\log_7 -3n = 4$

35) $\log_4 (6b + 4) = 0$

36) $\log_{11} (10v + 1) = -1$

37) $\log_5 (-10x + 4) = 4$

38) $\log_9 (7 - 6x) = -2$

39) $\log_2 (10 - 5a) = 3$

40) $\log_8 (3k - 1) = 1$

10.6

# Functions - Interest

An application of exponential functions is compound interest. When money is invested in an account (or given out on loan) a certain amount is added to the balance. This money added to the balance is called interest. Once that interest is added to the balance, it will earn more interest during the next compounding period. This idea of earning interest on interest is called compound interest. For example, if you invest \$100 at 10% interest compounded annually, after one year you will earn \$10 in interest, giving you a new balance of \$110. The next year you will earn another 10% or \$11, giving you a new balance of \$121. The third year you will earn another 10% or \$12.10, giving you a new balance of \$133.10. This pattern will continue each year until you close the account.

There are several ways interest can be paid. The first way, as described above, is compounded annually. In this model the interest is paid once per year. But interest can be compounded more often. Some common compounds include compounded semi-annually (twice per year), quarterly (four times per year, such as quarterly taxes), montly (12 times per year, such as a savings account), weekly (52 times per year), or even daily (365 times per year, such as some student loans). When interest is compounded in any of these ways we can calculate the balance after any amount of time using the following formula:

$$\begin{aligned} \textbf{Compound Interest Formula: } \boldsymbol{A} &= \boldsymbol{P\left(1+\frac{r}{n}\right)^{nt}} \\ A &= \text{Final Amount} \\ P &= \text{Principle (starting balance)} \\ r &= \text{Interest rate (as } a \text{ decimal)} \\ n &= \text{number of compounds per year} \\ t &= \text{time (in years)} \end{aligned}$$

**Example 41.**

If you take a car loan for \$25, 000 with an interest rate of 6.5% compounded quartly, no payments required for the first five years, what will your balance be at the end of those five years?

$$\begin{aligned} P = 25000, r = 0.065, n = 4, t = 5 & \quad \text{Identify each variable} \\ A = 25000\left(1+\frac{0.065}{4}\right)^{4\cdot 5} & \quad \text{Plug each value into formula, evaluate parenthesis} \\ A = 25000(1.01625)^{4\cdot 5} & \quad \text{Multiply exponents} \end{aligned}$$

$$
\begin{aligned}
A = 25000(1.01625)^{20} &\quad \text{Evaluate exponent} \\
A = 25000(1.38041977...) &\quad \text{Multiply} \\
A = 34510.49 & \\
\$34,510.49 &\quad \text{Our Solution}
\end{aligned}
$$

We can also find a missing part of the equation by using our techniques for solving equations.

**Example 42.**

What principle will amount to \$3000 if invested at 6.5% compounded weekly for 4 years?

$$
\begin{aligned}
A = 3000, r = 0.065, n = 52, t = 4 &\quad \text{Identify each variables} \\
3000 = P\left(1 + \frac{0.065}{52}\right)^{52\cdot 4} &\quad \text{Evaluate parentheses} \\
3000 = P(1.00125)^{52\cdot 4} &\quad \text{Multiply exponent} \\
3000 = P(1.00125)^{208} &\quad \text{Evaluate exponent} \\
3000 = P(1.296719528...) &\quad \text{Divide each side by } 1.296719528... \\
\overline{1.296719528...} \quad \overline{1.296719528...} & \\
2313.53 = P &\quad \text{Solution for } P \\
\$2313.53 &\quad \text{Our Solution}
\end{aligned}
$$

It is interesting to compare equal investments that are made at several differnt types of compounds. The next few examples do just that.

**Example 43.**

If \$4000 is invested in an account paying 3% interest compounded monthly, what is the balance after 7 years?

$$
\begin{aligned}
P = 4000, r = 0.03, n = 12, t = 7 &\quad \text{Idenfity each variable} \\
A = 4000\left(1 + \frac{0.03}{12}\right)^{12\cdot 7} &\quad \text{Plug each value into formula, evaluate parentheses} \\
A = 4000(1.0025)^{12\cdot 7} &\quad \text{Multiply exponents} \\
A = 4000(1.0025)^{84} &\quad \text{Evaluate exponent} \\
A = 4000(1.2333548) &\quad \text{Multiply} \\
A = 4933.42 & \\
\$4933.42 &\quad \text{Our Solution}
\end{aligned}
$$

To investigate what happens to the balance if the compounds happen more often, we will consider the same problem, this time with interest compounded daily.

**Example 44.**

If \$4000 is invested in an account paying 3% interest compounded daily, what is the balance after 7 years?

$$\begin{aligned}
P = 4000, r = 0.03, n = 365, t = 7 &\quad \text{Idenfity each variable}\\
A = 4000\left(1 + \frac{0.03}{365}\right)^{365\cdot 7} &\quad \text{Plug each value into formula, evaluate parenthesis}\\
A = 4000(1.00008219...)^{365\cdot 7} &\quad \text{Multiply exponent}\\
A = 4000(1.00008219...)^{2555} &\quad \text{Evaluate exponent}\\
A = 4000(1.23366741....) &\quad \text{Multiply}\\
A = 4934.67 &\\
\$4934.67 &\quad \text{Our Solution}
\end{aligned}$$

While this difference isn't very large, it is a bit higher. The table below shows the result for the same problem with different compounds.

| Compound | Balance |
|---|---|
| Annually | \$4919.50 |
| Semi-Annually | \$4927.02 |
| Quarterly | \$4930.85 |
| Monthly | \$4933.42 |
| Weekly | \$4934.41 |
| Daily | \$4934.67 |

As the table illustrates, the more often interest is compounded, the higher the final balance will be. The reason is because we are calculating compound interest or interest on interest. So once interest is paid into the account it will start earning interest for the next compound and thus giving a higher final balance. The next question one might consider is what is the maximum number of compounds possible? We actually have a way to calculate interest compounded an infinite number of times a year. This is when the interest is compounded continuously. When we see the word "continuously" we will know that we cannot use the first formula. Instead we will use the following formula:

$$\begin{aligned}
\textbf{Interst Compounded Continuously: } \boldsymbol{A = Pe^{rt}}\\
A = \text{Final Amount}\\
P = \text{Principle (starting balance)}\\
e = a \text{ constant approximately } 2.71828183....\\
r = \text{Interest rate (written as } a \text{ decimal)}\\
t = \text{time (years)}
\end{aligned}$$

The variable $e$ is a constant similar in idea to pi ($\pi$) in that it goes on forever without repeat or pattern, but just a pi ($\pi$) naturally occurs in sevearl geometry applications, so does $e$ appear in many exponential applications, continuous interest being one of them. If you have a scientific calculator you problably have an $e$ button (often using the 2nd or shift key, then hit ln) that will be useful in caculating interest compounded continuously.

**Example 45.**

If \$4000 is invested in an account paying 3% interest compounded continuously, what is the balance after 7 years?

$$\begin{array}{rl} P=4000, r=0.03, t=7 & \text{Identify each of the variables} \\ A=4000e^{0.03\cdot 7} & \text{Multiply exponent} \\ A=4000e^{0.21} & \text{Evaluate } e^{0.21} \\ A=4000(1.23367806...) & \text{Multiply} \\ A=4934.71 & \\ \$4934.71 & \text{Our Solution} \end{array}$$

Albert Einstein once said that the most powerful force in the universe is compound interest. Consider the following example, illustrating how powerful compound interest can be.

**Example 46.**

If you invest \$6.16 in an account paying 12% interest compounded continuously for 100 years, and that is all you have to leave your children as an inheritance, what will the final balance be that they will receive?

$$\begin{array}{rl} P=6.16, r=0.12, t=100 & \text{Identify each of the variables} \\ A=6.16e^{0.12\cdot 100} & \text{Multiply exponent} \\ A=6.16e^{12} & \text{Evaluate} \\ A=6.16(162,544.79) & \text{Multiply} \\ A=1,002,569.52 & \\ \$1,002,569.52 & \text{Our Solution} \end{array}$$

In 100 years that one time investment of \$6.16 investment grew to over one million dollars! That's the power of compound interest!

# Practice - Interest Rate Problems

**Solve**

1) Find each of the following:
   a. \$500 invested at 4% compounded annually for 10 years.
   b. \$600 invested at 6% compounded annually for 6 years.
   c. \$750 invested at 3% compounded annually for 8 years.
   d. \$1500 invested at 4% compounded semiannually for 7 years.
   e. \$900 invested at 6% compounded semiannually for 5 years.
   f. \$950 invested at 4% compounded semiannually for 12 years.
   g. \$2000 invested at 5% compounded quarterly for 6 years.
   h. \$2250 invested at 4% compounded quarterly for 9 years.
   i. \$3500 invested at 6% compounded quarterly for 12 years.

j. All of the above compunded continuously.

2) What principal will amount to \$2000 if invested at 4% interest compounded semiannually for 5 years?

3) What principal will amount to \$3500 if invested at 4% interest compounded quarterly for 5 years?

4) What principal will amount to \$3000 if invested at 3% interest compounded semiannually for 10 years?

5) What principal will amount to \$2500 if invested at 5% interest compounded semiannually for 7.5 years?

6) What principal will amount to \$1750 if invested at 3% interest compounded quarterly for 5 years?

7) A thousand dollars is left in a bank savings account drawing 7% interest, compounded quarterly for 10 years. What is the balance at the end of that time?

8) A thousand dollars is left in a credit union drawing 7% compounded monthly. What is the balance at the end of 10 years?

9) \$1750 is invested in an account earning 13.5% interest compounded monthly for a 2 year period.

10) You lend out \$5500 at 10% compounded monthly. If the debt is repaid in 18 months, what is the total owed at the time of repayment?

11) A $\$10,000$ Treasury Bill earned 16% compounded monthly. If the bill matured in 2 years, what was it worth at maturity?

12) You borrow $\$25,000$ at 12.25% interest compounded monthly. If you are unable to make any payments the first year, how much do you owe, excluding penalties?

13) A savings institution advertises 7% annual interest, compounded daily, How much more interest would you earn over the bank savings account or credit union in problems 7 and 8?

14) An 8.5% account earns continuous interest. If \$2500 is deposited for 5 years, what is the total accumulated?

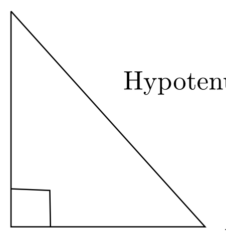
15) You lend \$100 at 10% continuous interest. If you are repaid 2 months later, what is owed?

10.7

# Functions - Trigonometry

There are six special functions that describe the relationship between the sides of a right triangle and the angles of the triangle. We will discuss three of the functions here. The three functions are called the sine, cosine, and tangent (the three others are cosecant, secant, and cotangent, but we will not need to use them here).

To the right is a picture of a right triangle. Based on which angle we are interested in on a given problem we will name the three sides in relationship to that angle. In the picture, angle A is the angle we will use to name the other sides. The longest side, the side opposite the right angle is always called the hypotenouse. The side across from the angle A is called the opposite side.




The third side, the side between our angle and the right angle is called the adjacent side. It is important to remember that the opposite and adjacent sides are named in relationship to the angle A or the angle we are using in a problem. If the angle had been the top angle, the opposite and adjacent sides would have been switched.

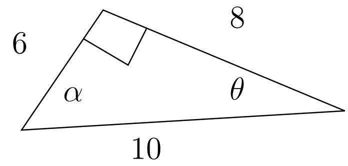
The three trigonometric funtions are functions taken of angles. When an angle goes into the function, the output is a ratio of two of the triangle sides. The ratios are as describe below:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

The "weird" variable $\theta$ is a greek letter, prounounced "theta" and is close in idea to our letter "t". Often working with triangles, the angles are repesented with Greek letters, in honor of the Ancient Greeks who developed much of Geometry. Some students remember the three ratios by remembering the word "SOH CAH TOA" where each letter if the first word of: "Sine: Opposite over Hypotenuse; Cosine: Adjacent over Hypotenuse; and Tangent: Opposite over Adjacent." Knowing how to use each of these relationships is fundamental to solving problems using trigonometry.

**Example 47.**

Using the diagram at right, find each of the following: sin$\theta$, cos$\theta$, tan$\theta$, sin$\alpha$, cos$\alpha$, and tan$\alpha$.

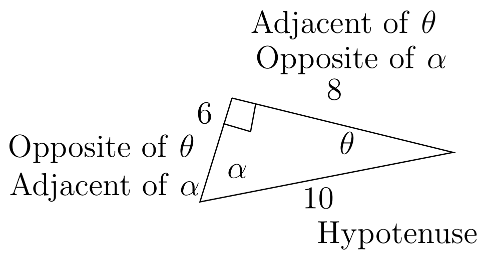



First we will find the three ratios of $\theta$. The hypotenuse is 10, from $\theta$, the opposite side is 6 and the adjacent side is 8. So we fill in the following:

$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{8}{10} = \frac{4}{5}$

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{6}{8} = \frac{3}{4}$




Now we will find the three ratios of $\alpha$. The hypotenuse is 10, from $\alpha$, the opposite side is 8 and the adjacent side is 6. So we fill in the following:

$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{8}{10} = \frac{4}{5}$

$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$

$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{8}{6} = \frac{4}{3}$

We can either use a trigonometry table or a calculator to find decimal values for sine, cosine, or tangent of any angle. We only put angle values into the trigonometric functions, never values for sides. Using either a table or a calculator, we can solve the next example.

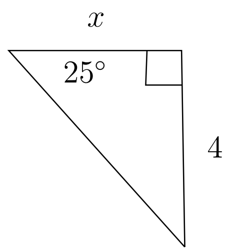
**Example 48.**

| | |
|---|---|
| $\sin 42°$ | Use calculator or table |
| $0.669$ | Our Solution |
| | |
| $\tan 12°$ | Use calculator or table |
| $0.213$ | Our Solution |
| | |
| $\cos 18°$ | Use calculator or table |
| $0.951$ | Our Solution |

By combining the ratios together with the decimal approximations the calculator or table gives us we can solve for missing sides of a triangle. The trick will be to determien which angle we are working with, naming the sides we are working with, and deciding which trig function can be used with the sides we have.

**Example 49.**

Find the measure of the missing side.




We will be using the angle marked 25°, from this angle, the side marked 4 is the opposite side and the side marked $x$ is the adjacent side.

The trig ratio that uses the opposite and adjacent sides is tangent. So we will take the tangent of our angle.

$$\tan 25° = \frac{4}{x} \quad \text{Tangent is opposite over adjacent}$$

$$\frac{0.466}{1} = \frac{4}{x} \quad \text{Evaluate } \tan 25 \circ \text{, put over 1 so we have proportion}$$

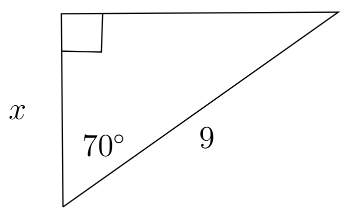
$$0.466x = 4 \quad \text{Find cross product}$$

$$\overline{0.466} \quad \overline{0.466} \quad \text{Divide both sides by } 0.466$$

$$x = 8.58 \quad \text{Our Solution}$$

**Example 50.**

Find the measure of the missing side.




We will be using the angle marked 70°. From this angle, the $x$ is the adjacent side and the 9 is the hypotenuse.

The trig ratio that uses adjacent and hypotenuse is the cosine. So we will take the cosine of our angle.

$$\cos 70° = \frac{x}{9} \quad \text{Cosine is adjacent over hypotenuse}$$

$$\frac{0.342}{1} = \frac{x}{9} \quad \text{Evaluate } \cos 70°\text{, put over 1 so we have } a \text{ proportion}$$

$$3.08 = 1x \quad \text{Find the cross product.}$$

$$3.08 = x \quad \text{Our Solution.}$$

# Practice - Trigonometry

**Find the value of each. Round your answers to the nearest ten-thousandth.**

1) $\cos 71°$

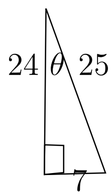
2) $\cos 23°$

3) $\sin 75°$

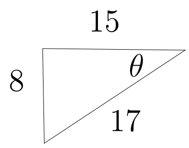
4) $\sin 50°$

**Find the value of the trig function indicated.**

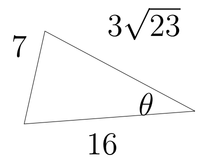
5) $\sin\ \theta$



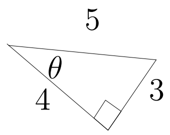

6) $\tan\ \theta$



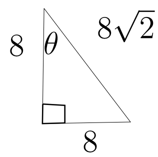

7) $\sin\ \theta$



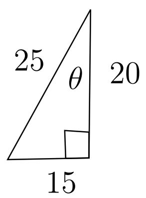

8) $\sin\ \theta$




9) $\sin\ \theta$

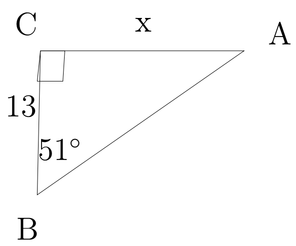


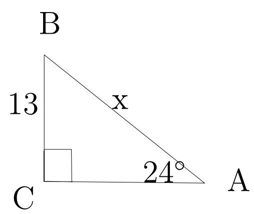

10) $\cos\ \theta$




**Find the measure of each side indicated. Round to the nearest tenth.**

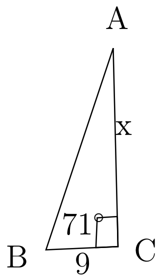
11)




13)

15)

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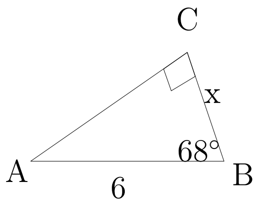


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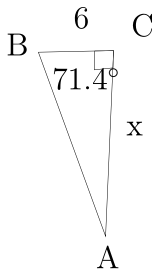


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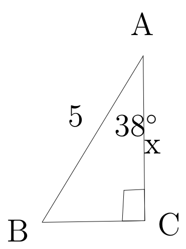


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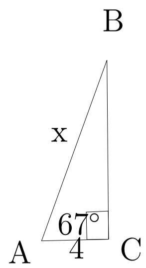


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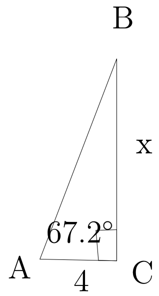


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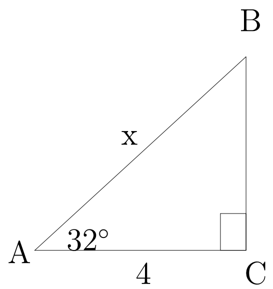


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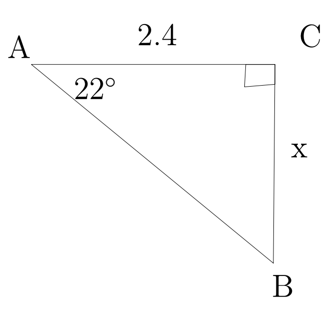
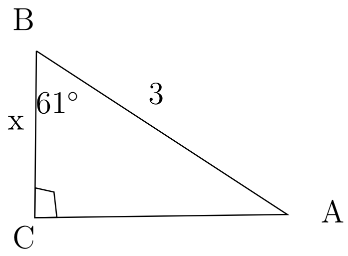
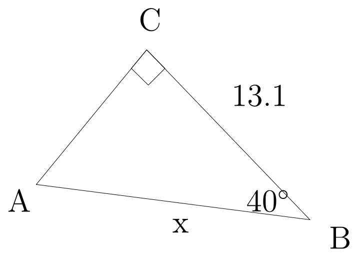


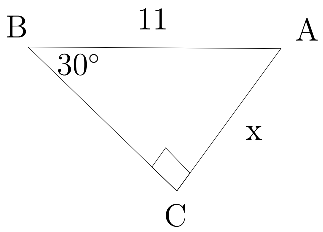

31)

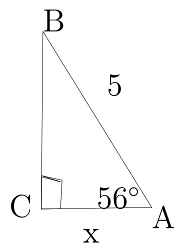



33)

12)

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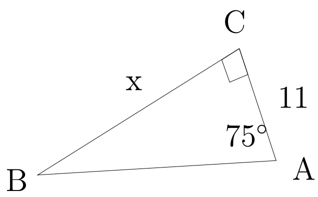


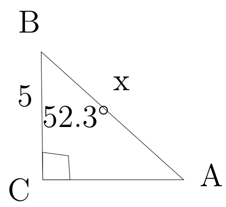


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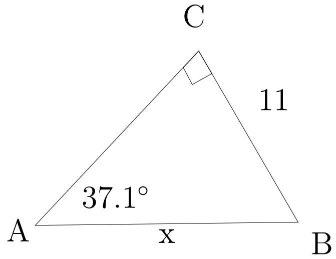


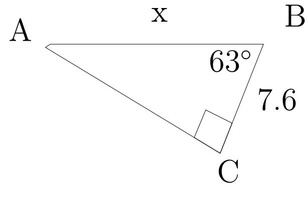


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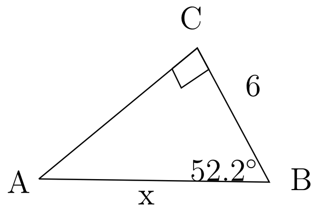


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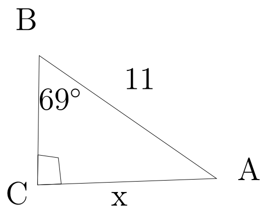


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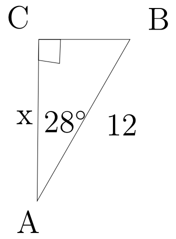


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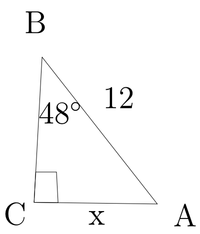


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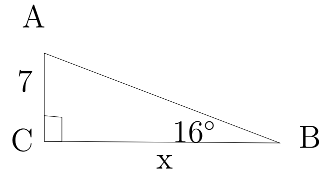


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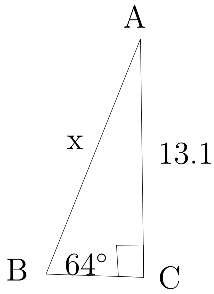


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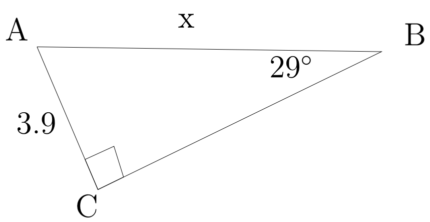


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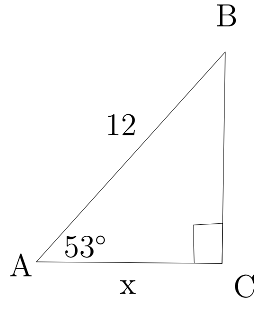


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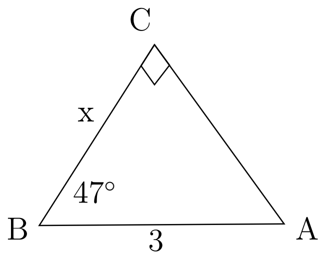


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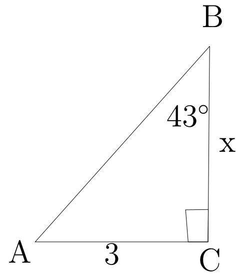


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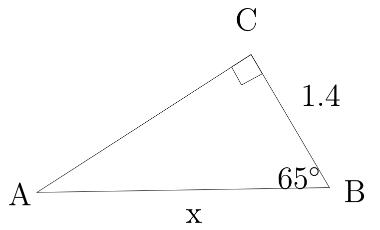


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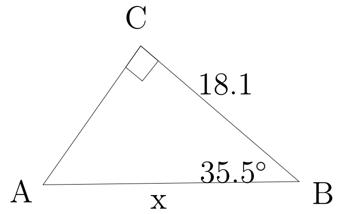


10.8

# Functions - Inverse Trigonometry

We used a special function, one of the trig functions, to take an angle of a triangle and find the side length. Here we will do the opposite, take the side lengths and find the angle. Because this is the opposite operation, we will use the inverse function of each of the trig ratios we saw before. The notation we will use for the inverse trig functions will be similar to the inverse notation we used with functions.

$$\mathbf{sin^{-1}}\left(\frac{\textbf{opposite}}{\textbf{hypotenuse}}\right)=\boldsymbol{\theta} \quad \mathbf{cos^{-1}}\left(\frac{\textbf{adjacent}}{\textbf{hypotenuse}}\right)=\boldsymbol{\theta} \quad \mathbf{tan^{-1}}\left(\frac{\textbf{opposite}}{\textbf{adjacent}}\right)=\boldsymbol{\theta}$$

Just as with inverse functions, the $-1$ is not an exponent, it is a notation to tell us that these are inverse functions. While the regular trig functions take angles as inputs, these inverse functions will always take a ratio of sides as inputs. We can calculate inverse trig values using a table or a calculator (usually pressing shift or 2nd first).

**Example 51.**

$$\begin{array}{rl}
\sin \mathrm{A}=0.5 & \text{We don't know the angle so we use an inverse trig function}\\
\sin^{-1}(0.5)=A & \text{Evaluate using table or calculator}\\
30^\circ=A & \text{Our Solution}
\end{array}$$

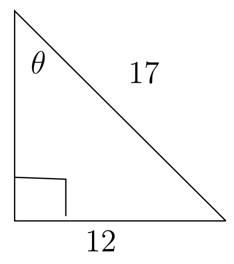
$$\begin{array}{rl}
\cos \mathrm{B}=0.667 & \text{We don't know the angle so we use an inverse trig function}\\
\cos^{-1}(0.667)=B & \text{Evaluate using table or calculator}\\
48^\circ=B & \text{Our Solution}
\end{array}$$

$$\begin{array}{rl}
\tan \mathrm{C}=1.54 & \text{We don't know the angle so we use an inverse trig function}\\
\tan^{-1}(1.54)=C & \text{Evaluate using table or calculator}\\
57^\circ=C & \text{Our Solution}
\end{array}$$

If we have two sides of a triangle, we can easily calculate their ratio as a decimal and then use one of the inverse trig functions to find a missing angle.

**Example 52.**

Find the indicated angle.




From angle $\theta$ the given sides are the opposite (12) and the hypotenuse (17).

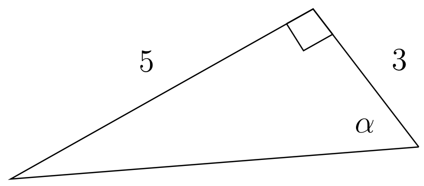
The trig functions that uses opposite and hypotenuse is the sine

Because we are lookin for an angle we use the inverse sine

$$
\begin{array}{rl}
\sin^{-1}\left(\dfrac{12}{17}\right) & \text{Sine is opposite over hyptenuse, use inverse to find angle} \\
\sin^{-1}(0.706) & \text{Evaluate fraction, take sine inverse using table or calculator} \\
45^\circ & \text{Our Solution}
\end{array}
$$

**Example 53.**

Find the indicated angle




From the angle $\alpha$, the given sides are the opposite (5) and the adjacent (3)

The trig function that uses opposite and adjacent is the tangent

As we are lookin for an angle we ill use the inverse tangent.

$$
\begin{array}{rl}
\tan^{-1}\left(\dfrac{5}{3}\right) & \text{Tangent is opposite over adjacent. Use inverse to find angle} \\
\tan^{-1}(1.667) & \text{Evaluate fraction, take tangent inverse on table or calculator} \\
50^\circ & \text{Our Solution}
\end{array}
$$

Using a combination of trig functions and inverse trig functions, if we are given two parts of a right triangle (two sides or a side and an angle), we can find all the other sides and angles of the triangle. This is called solving a triangle.

When we are solving a triangle, we can use trig ratios to solve for all the missing parts of it, but there are some properties from geometry that may be helpful along the way.

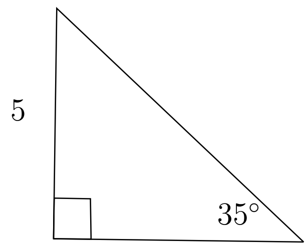
The angles of a triangle always add up to 180°, because we have a right triangle, 90° are used up in the right angle, that means there are another 90° left in the two acute angles. In other words, the smaller two angles will always add to 90, if we know one angle, we can quickly find the other by subtracting from 90.

Another trick is on the sides of the angles. If we know two sides of the right triangle, we can use the Pythagorean Theorem to find the third side. The Pythagorean Theorem states that if $c$ is the hypotenuse of the triangle, and $a$ and $b$ are the other two sides (legs), then we can use the following formula, $a^2 + b^2 = c^2$ to find a missing side.

Often when solving triangles we use trigonometry to find one part, then use the angle sum and/or the Pythagorean Theorem to find the other two parts.

**Example 54.**

Solve the triangle



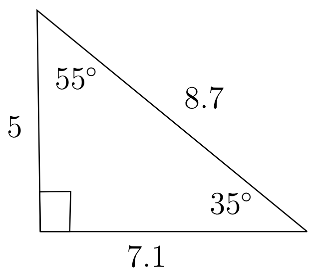

We have one angle and one side. We can use these to find either other side. We will find the other leg, the adjacent side to the 35° angle.

The 5 is the opposite side, so we will use the tangent to find the leg.

$$\begin{aligned} \tan 35^\circ &= \frac{5}{x} && \text{Tangent is opposite over adjacent} \\ \frac{0.700}{1} &= \frac{5}{x} && \text{Evaluate tangent, put it over one so we have } a \text{ proportion} \\ 0.700x &= \phantom{0.}5 && \text{Find cross product} \\ \overline{0.700} & \quad \overline{0.700} && \text{Divide both sides by } 0.700 \\ x &= 7.1 && \text{The missing leg.} \end{aligned}$$

$$\begin{aligned} a^2 + b^2 &= c^2 && \text{We can now use pythagorean thorem to find hypotenuse, } c \\ 5^2 + 7.1^2 &= c^2 && \text{Evaluate exponents} \\ 25 + 50.41 &= c^2 && \text{Add} \\ 75.41 &= c^2 && \text{Square root both sides} \\ 8.7 &= c && \text{The hypotenuse} \end{aligned}$$

$$\begin{aligned} 90^\circ - 35^\circ & && \text{To find the missing angle we subtract from } 90^\circ \\ 55^\circ & && \text{The missing angle} \end{aligned}$$

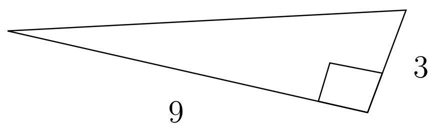



Our Solution

In the previous example, once we found the leg to be 7.1 we could have used the sine function on the 35° angle to get the hypotenuse and then any inverse trig function to find the missing angle and we would have found the same answers. The angle sum and pythagorean theorem are just nice shortcuts to solve the problem quicker.

**Example 55.**

Solve the triangle

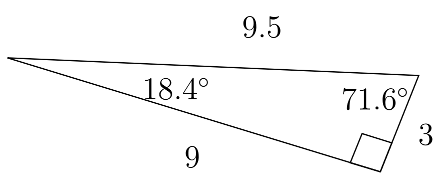



In this triangle we have two sides. We will first find the angle on the right side, adjacent to 3 and opposite from the 9.

Tangent uses opposite and adjacent

To find an angle we use the inverse tangent.

$$\begin{aligned} \tan^{-1}\left(\frac{9}{3}\right) &\quad \text{Evaluate fraction} \\ \tan^{-1}(3) &\quad \text{Evaluate tangent} \\ 71.6^\circ &\quad \text{The angle on the right side} \\ \\ 90^\circ - 71.6^\circ &\quad \text{Subtract angle from 90 to get other angle} \\ 18.4^\circ &\quad \text{The angle on the left side} \\ \\ a^2 + b^2 = c^2 &\quad \text{Pythagorean theorem to find hypotenuse} \\ 9^2 + 3^2 = c^2 &\quad \text{Evaluate exponents} \\ 81 + 9 = c^2 &\quad \text{Add} \\ 90 = c^2 &\quad \text{Square root both sides} \\ 3\sqrt{10} \text{ or } 9.5 = c &\quad \text{The hypotenuse} \end{aligned}$$




Our Solution

# Practice - Inverse Trigonometry

**Find each angle measure to the nearest degree.**

1) sin Z = 0.4848

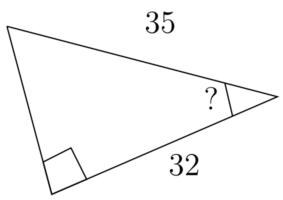
2) sin Y = 0.6293

3) sin Y = 0.6561

4) cos Y = 0.6157

**Find the measure of the indicated angle to the nearest degree.**

5)

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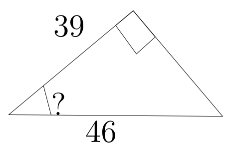


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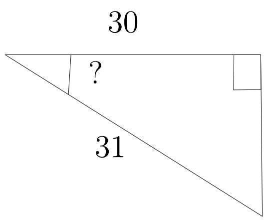


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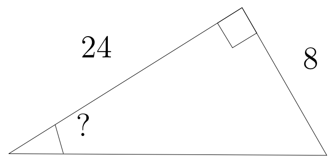


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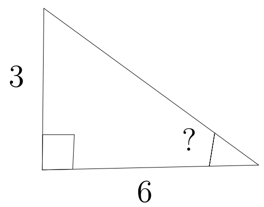


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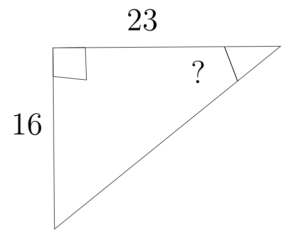
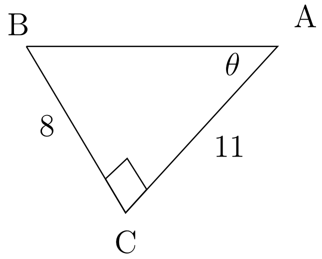



**Find the measure of each angle indicated. Round to the nearest tenth.**

11)

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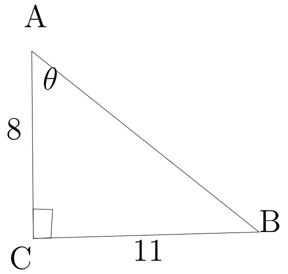


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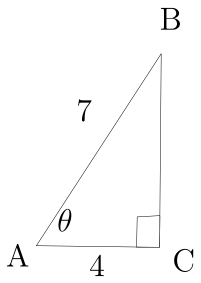


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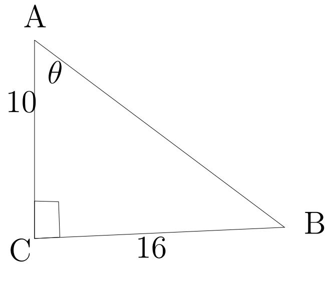


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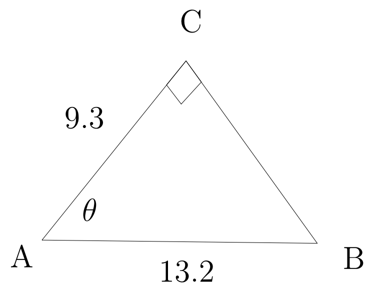


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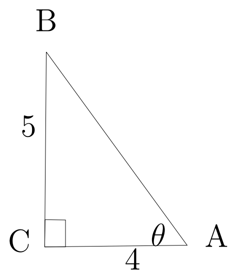


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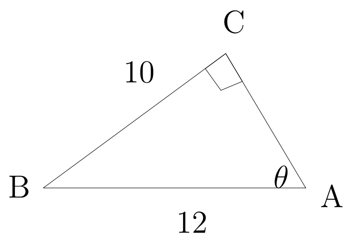


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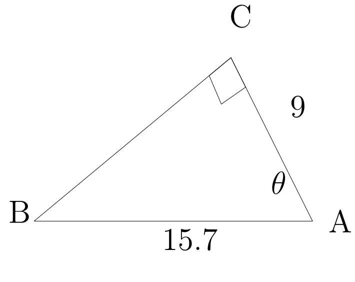


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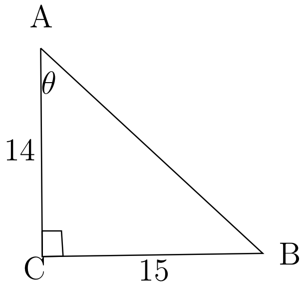


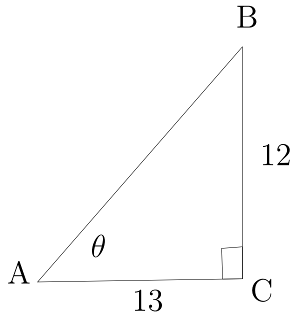


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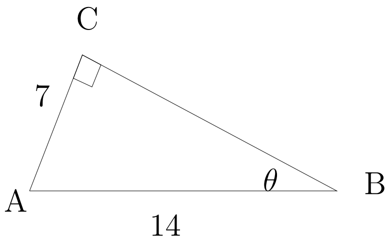


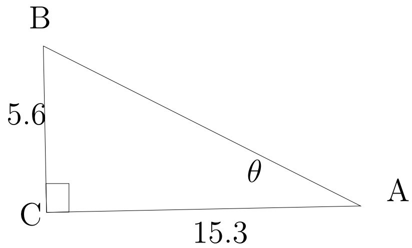


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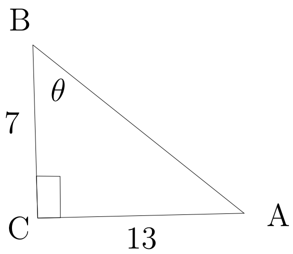


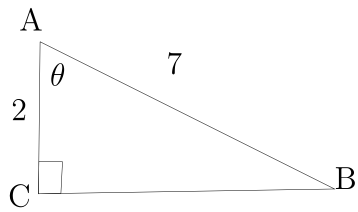


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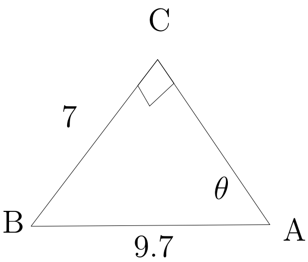


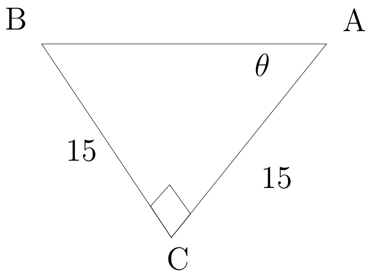


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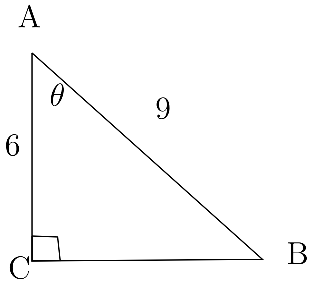


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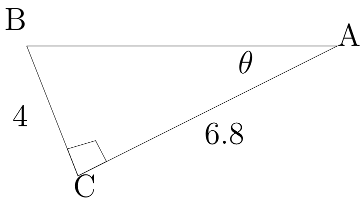


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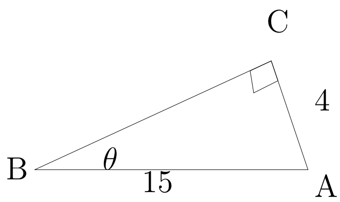


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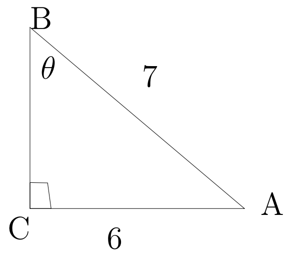
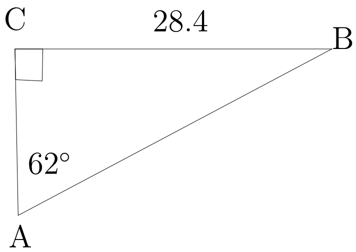



**Solve each triangle. Round answers to the nearest tenth.**

31)

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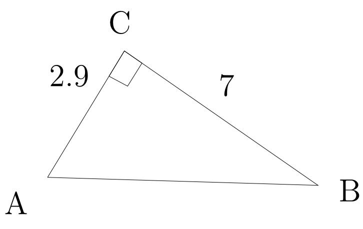


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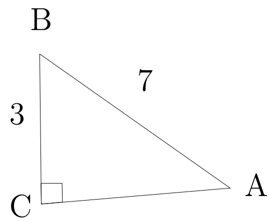


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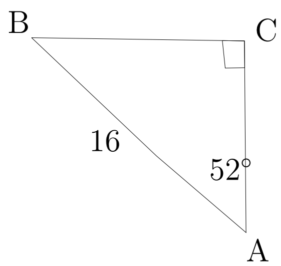


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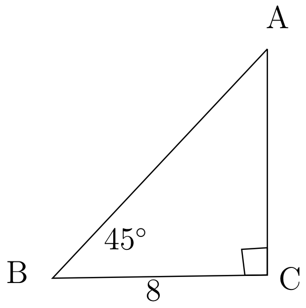


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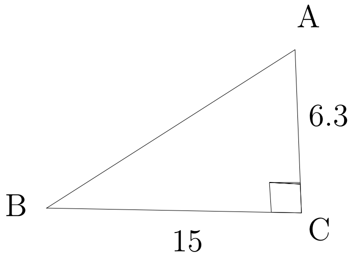
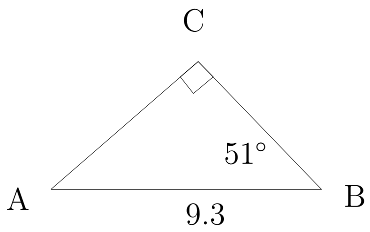
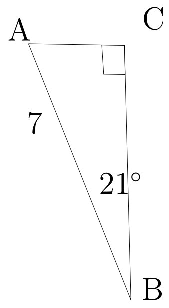


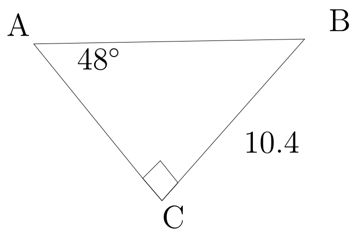

34)



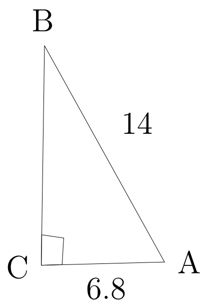

36)




38)




40)



## Answers - Function Notation

1) a. yes b. yes c. no d. no e. yes f. no g. yes h. no

2) all $x$

3) $x \leqslant \frac{5}{4}$

4) $t \neq 0$

5) all $x$

6) all $t$

7) $x \geqslant 16$

8) $x \neq -1, 4$

9) $x \geqslant 4, x \neq 5$

10) $x \neq \pm 5$

11) $-4$

12) $-\frac{3}{25}$

13) 2

14) 85

15) $-6$

16) 7

17) $-\frac{17}{9}$

18) $-6$

19) 13

20) 5

21) 11

22) $-21$

23) $\frac{1}{16}$

24) $-4$

25) $-21$

26) 2

27) $-60$

28) $-32$

29) 2

30) $\frac{31}{32}$

31) $-64x^3 + 2$

32) $4x + 10$

33) $-1 + 3x$

34) $-3 \cdot 2^{\frac{12+a}{4}}$

35) $2\left|-3n^2 - 1\right| + 2$

36) $1 + \frac{1}{16}x^2$

37) $3x + 1$

38) $t^4 + t^2$

39) $5^{-3-x}$

40) $5^{\frac{-2+n}{2}} + 1$

## Answers - Algebra of Functions

1) 82

2) 20

3) 46

4) 2

5) 5

6) $-30$

7) $-3$

8) 140

9) $-\frac{2}{3}$

10) $-43$

11) 100

12) $-74$

13) $\frac{1}{5}$

14) 27

15) $-\frac{9}{26}$

16) $n^2 - 2n$

17) $-x^3 - 4x - 2$

18) $-x^3 + 2x^2 - 3$

19) $-x^2 - 8x + 2$

20) $2t^2 - 8t$

21) $4x^3 + 25x^2 + 25x$

22) $-2t^3 - 15t^2 - 25t$

23) $x^2 - 4x + 5$

24) $3x^2 + 4x - 9$

25) $\frac{n^2 + 5}{3n + 5}$

26) $-2x + 9$

27) $\frac{-2a + 5}{3a + 5}$

28) $t^3 + 3t^2 - 3t + 5$

29) $n^3 + 8n + 5$

30) $\frac{4x+2}{x^2+2x}$

31) $n^6-9n^4+20n^2$

32) $18n^2-15n-25$

33) $x+3$

34) $-\frac{2}{3}$

35) $t^4+8t^2+2$

36) $\frac{3n-6}{-n^2-4n}$

37) $\frac{-x^3-2x}{-3x+4}$

38) $x^4-4x^2-3$

39) $\frac{-n^2-2n}{3}$

40) $\frac{32+32n-n^3}{8}$

41) $-155$

42) 5

43) 21

44) 4

45) 103

46) 12

47) 050

48) 112

49) 176

50) 147

51) $16x^2+12x-4$

52) $-8a+14$

53) $-8a+2$

54) $t$

55) $4x^3$

56) $-2n^2-12n-16$

57) $-2x+8$

58) $27t^3-108t^2+141t-60$

59) $-16t-5$

60) $3x^3+6x^2-4$

## Answers - Inverse Functions

1) Yes

2) No

3) Yes

4) Yes

5) No

6) Yes

7) No

8) Yes

9) Yes

10) No

11) $f^{-1}(x)=\sqrt[5]{x-3}-2$

12) $g^{-1}(x)=-1+(x-2)^3$

13) $g^{-1}(x)=\frac{4-2x}{x}$

14) $f^{-1}(x)=\frac{-3+3x}{x}$

15) $f^{-1}(x)=\frac{-2x-2}{x+2}$

16) $g^{-1}(x)=3x-9$

17) $f^{-1}(x)=-5x+10$

18) $f^{-1}(x)=\frac{15+2x}{5}$

19) $g^{-1}(x)=-\sqrt[3]{x}+1$

20) $f^{-1}(x)=\frac{-4x+12}{3}$

21) $f^{-1}(x)=\sqrt[3]{x}=3$

22) $g^{-1}(x)=-2x^5+2$

23) $g^{-1}(x)=\frac{x}{x-1}$

24) $f^{-1}(x)=\frac{-3x-3}{x+2}$

25) $f^{-1}(x)=\frac{-x-1}{x-1}$

26) $h^{-1}(x)=\frac{-2x}{x-1}$

27) $g^{-1}(x)=\frac{-4x+8}{5}$

28) $g^{-1}(x)=-3x+2$

29) $g^{-1}(x)=\frac{-x+1}{5}$

30) $f^{-1}(x)=\frac{5+4x}{5}$

31) $g^{-1}(x)=\sqrt[3]{x+1}$

32) $f^{-1}(x)=\sqrt[5]{\frac{-x+3}{2}}$

33) $h^{-1}(x)=-2(x-2)^3$

34) $g^{-1}(x)=\sqrt[3]{x-2}+1$

35) $f^{-1}(x)=\frac{-2x+1}{x-1}$

36) $f^{-1}(x)=\frac{-1-x}{x}$

37) $f^{-1}(x) = \frac{2x+7}{x+3}$

38) $f^{-1}(x) = -\frac{4x}{3}$

39) $g^{-1}(x) = -x$

40) $g^{-1}(x) = \frac{-3x+1}{2}$

Answers - Exponential Functions

1) $\{0\}$

2) $\{-1\}$

3) $\{0\}$

4) $\{0\}$

5) $\{-\frac{3}{4}\}$

6) $\{-\frac{5}{4}\}$

7) $\{-\frac{3}{2}\}$

8) $\{0\}$

9) $\{-\frac{2}{3}\}$

10) $\{0\}$

11) $\{\frac{5}{6}\}$

12) $\{0\}$

13) $\{-2\}$

14) $\{-\frac{5}{6}\}$

15) $\{1\}$

16) $\{-1\}$

17) No solution

18) $\{-\frac{4}{3}\}$

19) $\{-\frac{1}{4}\}$

20) $\{-\frac{3}{4}\}$

21) No solution

22) $\{0\}$

23) $\{-\frac{3}{2}\}$

24) $\{\frac{2}{5}\}$

25) $\{-1\}$

26) $\{\frac{1}{4}\}$

27) $\{-\frac{1}{2}\}$

28) $\{\frac{1}{3}\}$

29) $\{0\}$

30) No solution

31) $\{1\}$

32) $\{2\}$

33) $\{\frac{1}{3}\}$

34) $\{\frac{2}{3}\}$

35) $\{0\}$

36) $\{0\}$

37) $\{\frac{3}{8}\}$

38) $\{-\frac{3}{2}\}$

39) $\{-3\}$

40) No solution

Answers - Logarithmic Functions

1) $9^2 = 81$

2) $b^{-16} = a$

3) $7^{-2} = \frac{1}{49}$

4) $16^2 = 256$

5) $13^2 = 169$

6) $11^0 = 1$

7) $\log_8 1 = 0$

8) $\log_{17} \frac{1}{289} = -2$

9) $\log_{15} 225 = 2$

10) $\log_{144} 12 = \frac{1}{2}$

11) $\log_{64} 2 = \frac{1}{6}$

12) $\log_{19} 361 = 2$

13) $\frac{1}{3}$

14) 3

15) $-\frac{1}{3}$

16) 0

17) 2

18) $-3$

19) 2

20) $\frac{1}{2}$

21) 6

22) 6

23) $\{5\}$

24) $\{512\}$

25) $\{\frac{1}{4}\}$

26) $\{1000\}$

27) $\{121\}$

28) $\{256\}$

29) $\{6552\}$

30) $\{\frac{45}{11}\}$

31) $\{-\frac{125}{3}\}$

32) $\{-\frac{1}{4}\}$

33) $\{-\frac{54}{11}\}$

34) $\{-\frac{2401}{3}\}$

35) $\{-\frac{1}{2}\}$

36) $\{-\frac{1}{11}\}$

37) $\{-\frac{621}{10}\}$

38) $\{\frac{283}{243}\}$

39) $\{\frac{2}{45}\}$

40) $\{3\}$

## Answers - Interest Rate Problems

1)

a. 740.12; 745.91
b. 804.06; 809.92
c. 950.08; 953.44
d. 1979.22; 1984.69
e. 1209.52; 1214.87
f. 1528.02; 1535.27
g. 2694.70; 2699.72
h. 3219.23; 3224.99
i. 7152.17; 7190.52

2) 1640.70
3) 2868.41
4) 2227.41
5) 1726.16
6) 1507.08
7) 2001.60
8) 2009.66
9) 2288.98
10) 6386.12
11) 13742.19
12) 28240.43
13) 12.02; 3.96
14) 3823.98
15) 101.68

## Answers - Trigonometry

1) 0.3256
2) 0.9205
3) 0.9659
4) 0.7660
5) $\frac{7}{25}$
6) $\frac{8}{15}$
7) $\frac{7}{16}$
8) $\frac{3}{5}$
9) $\frac{\sqrt{2}}{2}$
10) $\frac{4}{5}$
11) 16.1
12) 2.8
13) 32
14) 8.2
15) 26.1
16) 16.8
17) 2.2
18) 9.8
19) 17.8
20) 10.3
21) 3.9
22) 10.6
23) 10.2
24) 8.9
25) 9.5
26) 24.4
27) 4.7
28) 14.6
29) 1
30) 8
31) 1.5
32) 7.2
33) 5.5
34)2

35) 41.1

36) 3.2

37) 18.2

38) 3.3

39) 17.1

40) 22.2

## Answers - Inverse Trigonometry

1) $29°$

2) $39°$

3) $41°$

4) $52°$

5) $24°$

6) $32°$

7) $15°$

8) $18°$

9) $27°$

10) $35°$

11) $36°$

12) $61.7°$

13) $54°$

14) $46.2°$

15) $55.2°$

16) $42.7°$

17) $58°$

18) $20.1°$

19) $45.2°$

20) $73.4°$

21) $51.3°$

22) $45°$

23) $56.4°$

24) $48.2°$

25) $55°$

26) $30.5°$

27) $47°$

28) $15.5°$

29) $30°$

30) $59°$

31) $m\angle B = 28°, b = 141, c = 32.2$

32) $m\angle B = 22.8°$, $m\angle A = 67.2°$, $c = 16.3$

33) $m\angle B = 22.5°, m\angle A = 67.5°, c = 7.6$

34) $m\angle A = 39°, b = 7.2, a = 5.9$

35) $m\angle B = 64.6°, m\angle A = 25.4°, b = 6.3$

36) $m\angle A = 69°, b = 2.5, a = 6.5$

37) $m\angle B = 38°, b = 9.9, a = 12.6$

38) $m\angle B = 42°, b = 9.4, c = 14$

39) $m\angle A = 45°, b = 8, c = 11.3$

40) $m\angle B = 29.1°$, $m\angle A = 60.9°$, $a = 12.2$