Radicals - Rationalize Denominators

Objective: Rationalize the denominators of radical expressions.

It is considered bad practice to have a radical in the denominator of a fraction. When this happens we multiply the numerator and denominator by the same thing in order to clear the radical. In the lesson on dividing radicals we talked about how this was done with monomials. Here we will look at how this is done with binomials.

If the binomial is in the numerator the process to rationalize the denominator is essentially the same as with monomials. The only difference is we will have to distribute in the numerator.

Example 1.

$$\frac{\sqrt{3}-9}{2\sqrt{6}} \quad \text{Want to clear } \sqrt{6} \text{ in denominator, multiply by } \sqrt{6}$$

$$\frac{(\sqrt{3}-9)}{2\sqrt{6}}\left(\frac{\sqrt{6}}{\sqrt{6}}\right) \quad \text{We will distribute the } \sqrt{6} \text{ through the numerator}$$

$$\frac{\sqrt{18}-9\sqrt{6}}{2\cdot 6}$$ Simplify radicals in numerator, multiply out denominator

$$\frac{\sqrt{9\cdot 2}-9\sqrt{6}}{12}$$ Take square root where possible

$$\frac{3\sqrt{2}-9\sqrt{6}}{12}$$ Reduce by dividing each term by 3

$$\frac{\sqrt{2}-3\sqrt{6}}{4}$$ Our Solution

It is important to remember that when reducing the fraction we cannot reduce with just the 3 and 12 or just the 9 and 12. When we have addition or subtraction in the numerator or denominator we must divide all terms by the same number.

The problem can often be made easier if we first simplify any radicals in the problem.

$$\frac{2\sqrt{20x^5}-\sqrt{12x^2}}{\sqrt{18x}}$$ Simplify radicals by finding perfect squares

$$\frac{2\sqrt{4\cdot 5x^3}-\sqrt{4\cdot 3x^2}}{\sqrt{9\cdot 2x}}$$ Simplify roots, divide exponents by 2.

$$\frac{2\cdot 2x^2\sqrt{5x}-2x\sqrt{3}}{3\sqrt{2x}}$$ Multiply coefficients

$$\frac{4x^2\sqrt{5x}-2x\sqrt{3}}{3\sqrt{2x}}$$ Multiplying numerator and denominator by $\sqrt{2x}$

$$\frac{(4x^2\sqrt{5x}-2x\sqrt{3})}{3\sqrt{2x}}\left(\frac{\sqrt{2x}}{\sqrt{2x}}\right)$$ Distribute through numerator

$$\frac{4x^2\sqrt{10x^2}-2x\sqrt{6x}}{3\cdot 2x}$$ Simplify roots in numerator,
multiply coefficients in denominator

$$\frac{4x^3\sqrt{10}-2x\sqrt{6x}}{6x}$$ Reduce, dividing each term by $2x$

$$\frac{2x^2\sqrt{10}-\sqrt{6x}}{3x} \quad \text{Our Solution}$$

As we are rationalizing it will always be important to constantly check our problem to see if it can be simplified more. We ask ourselves, can the fraction be reduced? Can the radicals be simplified? These steps may happen several times on our way to the solution.

If the binomial occurs in the denominator we will have to use a different strategy to clear the radical. Consider $\frac{2}{\sqrt{3}-5}$, if we were to multiply the denominator by $\sqrt{3}$ we would have to distribute it and we would end up with $3-5\sqrt{3}$. We have not cleared the radical, only moved it to another part of the denominator. So our current method will not work. Instead we will use what is called a conjugate. A **conjugate** is made up of the same terms, with the opposite sign in the middle. So for our example with $\sqrt{3}-5$ in the denominator, the conjugate would be $\sqrt{3}+5$. The advantage of a conjugate is when we multiply them together we have $(\sqrt{3}-5)(\sqrt{3}+5)$, which is a sum and a difference. We know when we multiply these we get a difference of squares. Squaring $\sqrt{3}$ and 5, with subtraction in the middle gives the product $3-25=-22$. Our answer when multiplying conjugates will no longer have a square root. This is exactly what we want.

Example 2.

$$\frac{2}{\sqrt{3}-5} \quad \text{Multiply numerator and denominator by conjugate}$$

$$\frac{2}{\sqrt{3}-5}\left(\frac{\sqrt{3}+5}{\sqrt{3}+5}\right) \quad \text{Distribute numerator, difference of squares in denominator}$$

$$\frac{2\sqrt{3}+10}{3-25} \quad \text{Simplify denoinator}$$

$$\frac{2\sqrt{3}+10}{-22} \quad \text{Reduce by dividing all terms by } -2$$

$$\frac{-\sqrt{3}-5}{11} \quad \text{Our Solution}$$

In the previous example, we could have reduced by dividng by 2, giving the solution $\frac{\sqrt{3}+5}{-11}$, both answers are correct.

Example 3.

$$\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}} \quad \text{Multiply by conjugate, } \sqrt{5}-\sqrt{3}$$

$$\frac{\sqrt{15}}{\sqrt{5}+\sqrt{3}}\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right) \quad \text{Distribute numerator, denominator is difference of squares}$$

$$\frac{\sqrt{75}-\sqrt{45}}{5-3} \quad \text{Simplify radicals in numerator, subtract in denominator}$$

$$\frac{\sqrt{25\cdot 3}-\sqrt{9\cdot 5}}{2} \quad \text{Take square roots where possible}$$

$$\frac{5\sqrt{3}-3\sqrt{5}}{2} \quad \text{Our Solution}$$

Example 4.

$$\frac{2\sqrt{3x}}{4-\sqrt{5x^3}} \quad \text{Multiply by conjugate, } 4+\sqrt{5x^3}$$

$$\frac{2\sqrt{3x}}{4-\sqrt{5x^3}}\left(\frac{4+\sqrt{5x^3}}{4+\sqrt{5x^3}}\right) \quad \text{Distribute numerator, denominator is difference of squares}$$

$$\frac{8\sqrt{3x}+2\sqrt{15x^4}}{16-5x^3} \quad \text{Simplify radicals where possible}$$

$$\frac{8\sqrt{3x}+2x^2\sqrt{15}}{16-5x^3} \quad \text{Our Solution}$$

The same process can be used when there is a binomial in the numerator and denominator. We just need to remember to FOIL out the numerator.

Example 5.

$$\frac{3-\sqrt{5}}{2-\sqrt{3}} \quad \text{Multiply by conjugate, } 2+\sqrt{3}$$

$$\frac{3-\sqrt{5}}{2-\sqrt{3}}\left(\frac{2+\sqrt{3}}{2+\sqrt{3}}\right) \quad \text{FOIL in numerator, denominator is difference of squares}$$

$$\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{4-3} \quad \text{Simplify denominator}$$

$$\frac{6+3\sqrt{3}-2\sqrt{5}-\sqrt{15}}{1} \quad \text{Divide each term by 1}$$

$$6+3\sqrt{3}-2\sqrt{5}-\sqrt{15} \quad \text{Our Solution}$$

Example 6.

$$\frac{2\sqrt{5}-3\sqrt{7}}{5\sqrt{6}+4\sqrt{2}} \quad \text{Multiply by the conjugate, } 5\sqrt{6}-4\sqrt{2}$$

$$\frac{2\sqrt{5}-3\sqrt{7}}{5\sqrt{6}+4\sqrt{2}}\left(\frac{5\sqrt{6}-4\sqrt{2}}{5\sqrt{6}-4\sqrt{2}}\right) \quad \begin{array}{l}\text{FOIL numerator,}\\ \text{denominator is difference of squares}\end{array}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}+12\sqrt{14}}{25\cdot 6-16\cdot 2} \quad \text{Multiply in denominator}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}+12\sqrt{14}}{150-32} \quad \text{Subtract in denominator}$$

$$\frac{10\sqrt{30}-8\sqrt{10}-15\sqrt{42}+12\sqrt{14}}{118} \quad \text{Our Solution}$$

The same process is used when we have variables

Example 7.

$$\frac{3x\sqrt{2x}+\sqrt{4x^3}}{5x-\sqrt{3x}} \quad \text{Multiply by the conjugate, } 5x+\sqrt{3x}$$

$$\frac{3x\sqrt{2x}+\sqrt{4x^3}}{5x-\sqrt{3x}}\left(\frac{5x+\sqrt{3x}}{5x+\sqrt{3x}}\right) \quad \begin{array}{l}\text{FOIL in numerator,}\\ \text{denominator is difference of squares}\end{array}$$

$$\frac{15x^2\sqrt{2x}+3x\sqrt{6x^2}+5x\sqrt{4x^3}+\sqrt{12x^4}}{25x^2-3x} \quad \text{Simplify radicals}$$

$$\frac{15x^2\sqrt{2x}+3x^2\sqrt{6}+10x^2\sqrt{x}+2x^2\sqrt{3}}{25x^2-3x} \quad \text{Divide each term by } x$$

$$\frac{15x\sqrt{2x}+3x\sqrt{6}+10x\sqrt{x}+2x\sqrt{3}}{25x-3} \quad \text{Our Solution}$$

World View Note: During the 5th century BC in India, Aryabhata published a treatise on astronomy. His work included a method for finding the square root of numbers that have many digits.

8.5 Practice - Rationalize Denominators

Simplify.

1) $\frac{4+2\sqrt{3}}{\sqrt{9}}$

2) $\frac{-4+\sqrt{3}}{4\sqrt{9}}$

3) $\frac{4+2\sqrt{3}}{5\sqrt{4}}$

4) $\frac{2\sqrt{3}-2}{2\sqrt{16}}$

5) $\frac{2-5\sqrt{5}}{4\sqrt{13}}$

6) $\frac{\sqrt{5}+4}{4\sqrt{17}}$

7) $\frac{\sqrt{2}-3\sqrt{3}}{\sqrt{3}}$

8) $\frac{\sqrt{5}-\sqrt{2}}{3\sqrt{6}}$

9) $\frac{5}{3\sqrt{5}+\sqrt{2}}$

10) $\frac{5}{\sqrt{3}+4\sqrt{5}}$

11) $\frac{2}{5+\sqrt{2}}$

12) $\frac{5}{2\sqrt{3}-\sqrt{2}}$

13) $\frac{3}{4-3\sqrt{3}}$

14) $\frac{4}{\sqrt{2}-2}$

15) $\frac{4}{3+\sqrt{5}}$

16) $\frac{2}{2\sqrt{5}+2\sqrt{3}}$

17) $-\frac{4}{4-4\sqrt{2}}$

18) $\frac{4}{4\sqrt{3}-\sqrt{5}}$

19) $\frac{1}{1+\sqrt{2}}$

20) $\frac{3+\sqrt{3}}{\sqrt{3}-1}$

21) $\frac{\sqrt{14}-2}{\sqrt{7}-\sqrt{2}}$

22) $\frac{2+\sqrt{10}}{\sqrt{2}+\sqrt{5}}$

23) $\frac{\sqrt{\mathrm{ab}}-a}{\sqrt{b}-\sqrt{a}}$

24) $\frac{\sqrt{14}-\sqrt{7}}{\sqrt{14}+\sqrt{7}}$

25) $\frac{a+\sqrt{\mathrm{ab}}}{\sqrt{a}+\sqrt{b}}$

26) $\frac{a+\sqrt{\mathrm{ab}}}{\sqrt{a}+\sqrt{b}}$

27) $\frac{2+\sqrt{6}}{2+\sqrt{3}}$

28) $\frac{2\sqrt{5}+\sqrt{3}}{1-\sqrt{3}}$

29) $\frac{a-\sqrt{b}}{a+\sqrt{b}}$

30) $\frac{a-b}{\sqrt{a}+\sqrt{b}}$

31) $\frac{6}{3\sqrt{2}-2\sqrt{3}}$

32) $\frac{\mathrm{ab}}{a\sqrt{b}-b\sqrt{a}}$

33) $\frac{a-b}{a\sqrt{b}-b\sqrt{a}}$

34) $\frac{4\sqrt{2}+3}{3\sqrt{2}+\sqrt{3}}$

35) $\frac{2-\sqrt{5}}{-3+\sqrt{5}}$

36) $\frac{-1+\sqrt{5}}{2\sqrt{5}+5\sqrt{2}}$

37) $\frac{5\sqrt{2}+\sqrt{3}}{5+5\sqrt{2}}$

38) $\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}-\sqrt{2}}$

Answers - Rationalize Denominators

1) $\frac{4+2\sqrt{3}}{3}$

2) $\frac{-4+\sqrt{3}}{12}$

3) $\frac{2+\sqrt{3}}{5}$

4) $\frac{\sqrt{3}-1}{4}$

5) $\frac{2\sqrt{13}-5\sqrt{65}}{52}$

6) $\frac{\sqrt{85}+4\sqrt{17}}{68}$

7) $\frac{\sqrt{6}-9}{3}$

8) $\frac{\sqrt{30}-2\sqrt{3}}{18}$

9) $\frac{15\sqrt{5}-5\sqrt{2}}{43}$

10) $\frac{-5\sqrt{3}+20\sqrt{5}}{77}$

11) $\frac{10-2\sqrt{2}}{23}$

12) $\frac{2\sqrt{3}+\sqrt{2}}{2}$

13) $\frac{-12-9\sqrt{3}}{11}$

14) $-2\sqrt{2}-4$

15) $3-\sqrt{5}$

16) $\frac{\sqrt{5}-\sqrt{3}}{2}$

17) $1+\sqrt{2}$

18) $\frac{16\sqrt{3}+4\sqrt{5}}{43}$

19) $\sqrt{2}-1$

20) $3+2\sqrt{3}$

21) $\sqrt{2}$

22) $\sqrt{2}$

23) $\sqrt{a}$

24) $3-2\sqrt{2}$

25) $\sqrt{a}$

26) $\frac{1}{3}$

27) $4-2\sqrt{3}+2\sqrt{6}-3\sqrt{2}$

28) $\frac{2\sqrt{5}-2\sqrt{15}+\sqrt{3}+3}{-2}$

29) $\frac{a^2-2a\sqrt{b}+b}{a^2-b}$

30) $\sqrt{a}-\sqrt{b}$

31) $3\sqrt{2}+2\sqrt{3}$

32) $\frac{a\sqrt{b}+b\sqrt{a}}{a-b}$

33) $\frac{a\sqrt{b}+b\sqrt{a}}{\text{ab}}$

34) $\frac{24-4\sqrt{6}+9\sqrt{2}-3\sqrt{3}}{15}$

35) $\frac{-1+\sqrt{5}}{4}$

36) $\frac{2\sqrt{5}-5\sqrt{2}-10+5\sqrt{10}}{30}$

37) $\frac{-5\sqrt{2}+10-\sqrt{3}+\sqrt{6}}{5}$

38) $\frac{8+3\sqrt{6}}{10}$