Functions - Interest

An application of exponential functions is compound interest. When money is invested in an account (or given out on loan) a certain amount is added to the balance. This money added to the balance is called interest. Once that interest is added to the balance, it will earn more interest during the next compounding period. This idea of earning interest on interest is called compound interest. For example, if you invest \$100 at 10% interest compounded annually, after one year you will earn \$10 in interest, giving you a new balance of \$110. The next year you will earn another 10% or \$11, giving you a new balance of \$121. The third year you will earn another 10% or \$12.10, giving you a new balance of \$133.10. This pattern will continue each year until you close the account.

There are several ways interest can be paid. The first way, as described above, is compounded annually. In this model the interest is paid once per year. But interest can be compounded more often. Some common compounds include compounded semi-annually (twice per year), quarterly (four times per year, such as quarterly taxes), montly (12 times per year, such as a savings account), weekly (52 times per year), or even daily (365 times per year, such as some student loans). When interest is compounded in any of these ways we can calculate the balance after any amount of time using the following formula:

$$\textbf{Compound Interest Formula: } \boldsymbol{A = P\left(1+\frac{r}{n}\right)^{nt}}$$

$$\begin{aligned} A &= \text{Final Amount} \\ P &= \text{Principle (starting balance)} \\ r &= \text{Interest rate (as } a \text{ decimal)} \\ n &= \text{number of compounds per year} \\ t &= \text{time (in years)} \end{aligned}$$

Example 1.

If you take a car loan for \$25, 000 with an interest rate of 6.5% compounded quartly, no payments required for the first five years, what will your balance be at the end of those five years?

$$\begin{aligned} P = 25000, r = 0.065, n = 4, t = 5 \quad & \text{Identify each variable} \\ A = 25000\left(1+\frac{0.065}{4}\right)^{4\cdot 5} \quad & \text{Plug each value into formula, evaluate parenthesis} \\ A = 25000(1.01625)^{4\cdot 5} \quad & \text{Multiply exponents} \end{aligned}$$

$$
\begin{aligned}
A = 25000(1.01625)^{20} &\quad \text{Evaluate exponent}\\
A = 25000(1.38041977...) &\quad \text{Multiply}\\
A = 34510.49 &\\
\$34,510.49 &\quad \text{Our Solution}
\end{aligned}
$$

We can also find a missing part of the equation by using our techniques for solving equations.

Example 2.

What principle will amount to \$3000 if invested at 6.5% compounded weekly for 4 years?

$$
\begin{aligned}
A = 3000, r = 0.065, n = 52, t = 4 &\quad \text{Identify each variables}\\
3000 = P\left(1 + \frac{0.065}{52}\right)^{52\cdot 4} &\quad \text{Evaluate parentheses}\\
3000 = P(1.00125)^{52\cdot 4} &\quad \text{Multiply exponent}\\
3000 = P(1.00125)^{208} &\quad \text{Evaluate exponent}\\
3000 = P(1.296719528...) &\quad \text{Divide each side by } 1.296719528...\\
\overline{1.296719528...} \quad \overline{1.296719528...} &\\
2313.53 = P &\quad \text{Solution for } P\\
\$2313.53 &\quad \text{Our Solution}
\end{aligned}
$$

It is interesting to compare equal investments that are made at several differnt types of compounds. The next few examples do just that.

Example 3.

If \$4000 is invested in an account paying 3% interest compounded monthly, what is the balance after 7 years?

$$
\begin{aligned}
P = 4000, r = 0.03, n = 12, t = 7 &\quad \text{Idenfity each variable}\\
A = 4000\left(1 + \frac{0.03}{12}\right)^{12\cdot 7} &\quad \text{Plug each value into formula, evaluate parentheses}\\
A = 4000(1.0025)^{12\cdot 7} &\quad \text{Multiply exponents}\\
A = 4000(1.0025)^{84} &\quad \text{Evaluate exponent}\\
A = 4000(1.2333548) &\quad \text{Multiply}\\
A = 4933.42 &\\
\$4933.42 &\quad \text{Our Solution}
\end{aligned}
$$

To investigate what happens to the balance if the compounds happen more often, we will consider the same problem, this time with interest compounded daily.

Example 4.

If \$4000 is invested in an account paying 3% interest compounded daily, what is the balance after 7 years?

$$
\begin{aligned}
P = 4000, r = 0.03, n = 365, t = 7 &\quad \text{Idenfity each variable} \\
A = 4000\left(1 + \frac{0.03}{365}\right)^{365\cdot 7} &\quad \text{Plug each value into formula, evaluate parenthesis} \\
A = 4000(1.00008219...)^{365\cdot 7} &\quad \text{Multiply exponent} \\
A = 4000(1.00008219...)^{2555} &\quad \text{Evaluate exponent} \\
A = 4000(1.23366741....) &\quad \text{Multiply} \\
A = 4934.67 & \\
\$4934.67 &\quad \text{Our Solution}
\end{aligned}
$$

While this difference isn't very large, it is a bit higher. The table below shows the result for the same problem with different compounds.

Compound	Balance
Annually	\$4919.50
Semi-Annually	\$4927.02
Quarterly	\$4930.85
Monthly	\$4933.42
Weekly	\$4934.41
Daily	\$4934.67

As the table illustrates, the more often interest is compounded, the higher the final balance will be. The reason is because we are calculating compound interest or interest on interest. So once interest is paid into the account it will start earning interest for the next compound and thus giving a higher final balance. The next question one might consider is what is the maximum number of compounds possible? We actually have a way to calculate interest compounded an infinite number of times a year. This is when the interest is compounded continuously. When we see the word "continuously" we will know that we cannot use the first formula. Instead we will use the following formula:

Interst Compounded Continuously: $\boldsymbol{A = Pe^{rt}}$

$$
\begin{aligned}
A &= \text{Final Amount} \\
P &= \text{Principle (starting balance)} \\
e &= a \text{ constant approximately } 2.71828183.... \\
r &= \text{Interest rate (written as } a \text{ decimal)} \\
t &= \text{time (years)}
\end{aligned}
$$

The variable e is a constant similar in idea to pi (π) in that it goes on forever without repeat or pattern, but just a pi (π) naturally occurs in sevearl geometry applications, so does e appear in many exponential applications, continuous interest being one of them. If you have a scientific calculator you problably have an e button (often using the 2nd or shift key, then hit ln) that will be useful in caculating interest compounded continuously.

Example 5.

If \$4000 is invested in an account paying 3% interest compounded continuously, what is the balance after 7 years?

$$
\begin{array}{rl}
P=4000, r=0.03, t=7 & \text{Identify each of the variables} \\
A=4000e^{0.03\cdot 7} & \text{Multiply exponent} \\
A=4000e^{0.21} & \text{Evaluate } e^{0.21} \\
A=4000(1.23367806...) & \text{Multiply} \\
A=4934.71 & \\
\$4934.71 & \text{Our Solution}
\end{array}
$$

Albert Einstein once said that the most powerful force in the universe is compound interest. Consider the following example, illustrating how powerful compound interest can be.

Example 6.

If you invest \$6.16 in an account paying 12% interest compounded continuously for 100 years, and that is all you have to leave your children as an inheritance, what will the final balance be that they will receive?

$$
\begin{array}{rl}
P=6.16, r=0.12, t=100 & \text{Identify each of the variables} \\
A=6.16e^{0.12\cdot 100} & \text{Multiply exponent} \\
A=6.16e^{12} & \text{Evaluate} \\
A=6.16(162,544.79) & \text{Multiply} \\
A=1,002,569.52 & \\
\$1,002,569.52 & \text{Our Solution}
\end{array}
$$

In 100 years that one time investment of \$6.16 investment grew to over one million dollars! That's the power of compound interest!

Practice - Interest Rate Problems

Solve

1) Find each of the following:
   a. \$500 invested at 4% compounded annually for 10 years.
   b. \$600 invested at 6% compounded annually for 6 years.
   c. \$750 invested at 3% compounded annually for 8 years.
   d. \$1500 invested at 4% compounded semiannually for 7 years.
   e. \$900 invested at 6% compounded semiannually for 5 years.
   f. \$950 invested at 4% compounded semiannually for 12 years.
   g. \$2000 invested at 5% compounded quarterly for 6 years.
   h. \$2250 invested at 4% compounded quarterly for 9 years.
   i. \$3500 invested at 6% compounded quarterly for 12 years.
   j. All of the above compunded continuously.

2) What principal will amount to \$2000 if invested at 4% interest compounded semiannually for 5 years?

3) What principal will amount to \$3500 if invested at 4% interest compounded quarterly for 5 years?

4) What principal will amount to \$3000 if invested at 3% interest compounded semiannually for 10 years?

5) What principal will amount to \$2500 if invested at 5% interest compounded semiannually for 7.5 years?

6) What principal will amount to \$1750 if invested at 3% interest compounded quarterly for 5 years?

7) A thousand dollars is left in a bank savings account drawing 7% interest, compounded quarterly for 10 years. What is the balance at the end of that time?

8) A thousand dollars is left in a credit union drawing 7% compounded monthly. What is the balance at the end of 10 years?

9) \$1750 is invested in an account earning 13.5% interest compounded monthly for a 2 year period.

10) You lend out \$5500 at 10% compounded monthly. If the debt is repaid in 18 months, what is the total owed at the time of repayment?

11) A \$10,000 Treasury Bill earned 16% compounded monthly. If the bill matured in 2 years, what was it worth at maturity?

12) You borrow \$25,000 at 12.25% interest compounded monthly. If you are unable to make any payments the first year, how much do you owe, excluding penalties?

13) A savings institution advertises 7% annual interest, compounded daily, How much more interest would you earn over the bank savings account or credit union in problems 7 and 8?

14) An 8.5% account earns continuous interest. If \$2500 is deposited for 5 years, what is the total accumulated?

15) You lend \$100 at 10% continuous interest. If you are repaid 2 months later, what is owed?

Answers - Interest Rate Problems

1)

a. 740.12; 745.91
b. 804.06; 809.92
c. 950.08; 953.44
d. 1979.22; 1984.69
e. 1209.52; 1214.87
f. 1528.02; 1535.27
g. 2694.70; 2699.72
h. 3219.23; 3224.99
i. 7152.17; 7190.52

2) 1640.70
3) 2868.41
4) 2227.41
5) 1726.16
6) 1507.08
7) 2001.60
8) 2009.66
9) 2288.98
10) 6386.12
11) 13742.19
12) 28240.43
13) 12.02; 3.96
14) 3823.98
15) 101.68