Functions - Operations on Functions

Objective: Combine functions using sum, difference, product, quotient and composition of functions.

Several functions can work together in one larger function. There are 5 common operations that can be performed on functions. The four basic operations on functions are adding, subtracting, multiplying, and dividing. The notation for these functions is as follows.

$$\begin{array}{rl} \text{Addition} & (f+g)(x)=f(x)+g(x) \\ \text{Subtraction} & (f-g)(x)=f(x)-g(x) \\ \text{Multiplication} & (f\cdot g)(x)=f(x)g(x) \\ \text{Division} & \left(\dfrac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)} \end{array}$$

When we do one of these four basic operations we can simply evaluate the two functions at the value and then do the operation with both solutions

Example 1.

$$\begin{array}{ll} f(x)=x^2-x-2 & \\ g(x)=x+1 & \text{Evaluate } f \text{ and } g \text{ at } -3 \\ \text{find } (f+g)(-3) & \end{array}$$

$$\begin{array}{rl} f(-3)=(-3)^2-(-3)-2 & \text{Evaluate } f \text{ at } -3 \\ f(-3)=9+3-2 & \\ f(-3)=10 & \end{array}$$

$$\begin{array}{rl} g(-3)=(-3)+1 & \text{Evaluate } g \text{ at } -3 \\ g(-3)=-2 & \end{array}$$

$$\begin{array}{rl} f(-3)+g(-3) & \text{Add the two functions together} \\ (10)+(-2) & \text{Add} \\ 8 & \text{Our Solution} \end{array}$$

The process is the same regardless of the operation being performed.

Example 2.

$$\begin{array}{ll} h(x)=2x-4 & \\ k(x)=-3x+1 & \text{Evaluate } h \text{ and } k \text{ at } 5 \\ \text{Find } (h\cdot k)(5) & \end{array}$$

$$\begin{array}{rl} h(5)=2(5)-4 & \text{Evaluate } h \text{ at } 5 \end{array}$$

$$h(5) = 10 - 4$$
$$h(5) = 6$$

$$k(5) = -3(5) + 1 \quad \text{Evaluate } k \text{ at } 5$$
$$k(5) = -15 + 1$$
$$k(5) = -14$$

$$h(5)k(5) \quad \text{Multiply the two results together}$$
$$(6)(-14) \quad \text{Multiply}$$
$$-84 \quad \text{Our Solution}$$

Often as we add, subtract, multiply, or divide functions, we do so in a way that keeps the variable. If there is no number to plug into the equations we will simply use each equation, in parenthesis, and simplify the expression.

Example 3.

$$f(x) = 2x - 4$$
$$g(x) = x^2 - x + 5 \quad \text{Write subtraction problem of functions}$$
$$\text{Find } (f - g)(x)$$

$$f(x) - g(x) \quad \text{Replace } f(x) \text{ with } (2x - 3) \text{ and } g(x) \text{ with } (x^2 - x + 5)$$
$$(2x - 4) - (x^2 - x + 5) \quad \text{Distribute the negative}$$
$$2x - 4 - x^2 + x - 5 \quad \text{Combine like terms}$$
$$-x^2 + 3x - 9 \quad \text{Our Solution}$$

The parenthesis are very important when we are replacing $f(x)$ and $g(x)$ with a variable. In the previous example we needed the parenthesis to know to distribute the negative.

Example 4.

$$f(x) = x^2 - 4x - 5$$
$$g(x) = x - 5 \quad \text{Write division problem of functions}$$
$$\text{Find } \left(\frac{f}{g}\right)(x)$$

$$\frac{f(x)}{g(x)} \quad \text{Replace } f(x) \text{ with } (x^2 - 4x - 5) \text{ and } g(x) \text{ with } (x - 5)$$

$$\frac{(x^2 - 4x - 5)}{(x - 5)} \quad \text{To simplify the fraction we must first factor}$$

$$\frac{(x-5)(x+1)}{(x-5)} \quad \text{Divide out common factor of } x-5$$

$$x+1 \quad \text{Our Solution}$$

Just as we could substitute an expression into evaluating functions, we can substitute an expression into the operations on functions.

Example 5.

$$\begin{aligned} f(x) &= 2x-1 \\ g(x) &= x+4 \quad && \text{Write as } a \text{ sum of functions} \\ \text{Find } (f+g)(x^2) \end{aligned}$$

$$\begin{aligned} f(x^2)+g(x^2) \quad & \text{Replace } x \text{ in } f(x) \text{ and } g(x) \text{ with } x^2 \\ [2(x^2)-1]+[(x^2)+4] \quad & \text{Distribute the } + \text{ does not change the problem} \\ 2x^2-1+x^2+4 \quad & \text{Combine like terms} \\ 3x^2+3 \quad & \text{Our Solution} \end{aligned}$$

Example 6.

$$\begin{aligned} f(x) &= 2x-1 \\ g(x) &= x+4 \quad && \text{Write as } a \text{ product of functions} \\ \text{Find } (f\cdot g)(3x) \end{aligned}$$

$$\begin{aligned} f(3x)g(3x) \quad & \text{Replace } x \text{ in } f(x) \text{ and } g(x) \text{ with } 3x \\ [2(3x)-1][(3x)+4] \quad & \text{Multiply our } 2(3x) \\ (6x-1)(3x+4) \quad & \text{FOIL} \\ 18x^2+24x-3x-4 \quad & \text{Combine like terms} \\ 18x^2+21x-4 \quad & \text{Our Solution} \end{aligned}$$

The fifth operation of functions is called composition of functions. A composition of functions is a function inside of a function. The notation used for composition of functions is:

$$\boldsymbol{(f\circ g)(x) = f(g(x))}$$

To calculate a composition of function we will evaluate the inner function and substitute the answer into the outer function. This is shown in the following example.

Example 7.

$$
\begin{aligned}
a(x) &= x^2 - 2x + 1 && \\
b(x) &= x - 5 && \text{Rewrite as } a \text{ function in function} \\
\text{Find } (a \circ b)(3) & &&
\end{aligned}
$$

$$
\begin{aligned}
a(b(3)) & && \text{Evaluate the inner function first, } b(3) \\
b(3) = (3) - 5 &= -2 && \text{This solution is put into } a, a(-2) \\
a(-2) &= (-2)^2 - 2(-2) + 1 && \text{Evaluate} \\
a(-2) &= 4 + 4 + 1 && \text{Add} \\
a(-2) &= 9 && \text{Our Solution}
\end{aligned}
$$

We can also evaluate a composition of functions at a variable. In these problems we will take the inside function and substitute into the outside function.

Example 8.

$f(x) = x^2 - x$	
$g(x) = x + 3$	Rewrite as a function in function
Find $(f \circ g)(x)$	

$f(g(x))$	Replace $g(x)$ with $x + 3$
$f(x + 3)$	Replace the variables in f with $(x + 3)$
$(x + 3)^2 - (x + 3)$	Evaluate exponent
$(x^2 + 6x + 9) - (x + 3)$	Distribute negative
$x^2 + 6x + 9 - x - 3$	Combine like terms
$x^2 + 5x + 6$	Our Solution

It is important to note that very rarely is $(f \circ g)(x)$ the same as $(g \circ f)(x)$ as the following example will show, using the same equations, but compositing them in the opposite direction.

Example 9.

$f(x) = x^2 - x$	
$g(x) = x + 3$	Rewrite as a function in function
Find $(g \circ f)(x)$	

$g(f(x))$	Replace $f(x)$ with $x^2 - x$
$g(x^2 - x)$	Replace the variable in g with $(x^2 - x)$
$(x^2 - x) + 3$	Here the parenthesis don$'t$ change the expression
$x^2 - x + 3$	Our Solution

World View Note: The term “function” came from Gottfried Wihelm Leibniz, a German mathematician from the late 17th century.

10.2 Practice - Operations on Functions

Perform the indicated operations.

1) $g(a) = a^3 + 5a^2$
$f(a) = 2a + 4$
Find $g(3) + f(3)$

2) $f(x) = -3x^2 + 3x$
$g(x) = 2x + 5$
Find $f(-4) \div g(-4)$

3) $g(a) = 3a + 3$
$f(a) = 2a - 2$
Find $(g + f)(9)$

4) $g(x) = 4x + 3$
$h(x) = x^3 - 2x^2$
Find $(g - h)(-1)$

5) $g(x) = x + 3$
$f(x) = -x + 4$
Find $(g - f)(3)$

6) $g(x) = -4x + 1$
$h(x) = -2x - 1$
Find $g(5) + h(5)$

7) $g(x) = x^2 + 2$
$f(x) = 2x + 5$
Find $(g - f)(0)$

8) $g(x) = 3x + 1$
$f(x) = x^3 + 3x^2$
Find $g(2) \cdot f(2)$

9) $g(t) = t - 3$
$h(t) = -3t^3 + 6t$
Find $g(1) + h(1)$

10) $f(n) = n - 5$
$g(n) = 4n + 2$
Find $(f + g)(-8)$

11) $h(t) = t + 5$
$g(t) = 3t - 5$
Find $(h \cdot g)(5)$

12) $g(a) = 3a - 2$
$h(a) = 4a - 2$
Find $(g + h)(-10)$

13) $h(n) = 2n - 1$
$g(n) = 3n - 5$
Find $h(0) \div g(0)$

14) $g(x) = x^2 - 2$
$h(x) = 2x + 5$
Find $g(-6) + h(-6)$

15) $f(a) = -2a - 4$
$g(a) = a^2 + 3$
Find $(\frac{f}{g})(7)$

16) $g(n) = n^2 - 3$
$h(n) = 2n - 3$
Find $(g - h)(n)$

17) $g(x) = -x^3 - 2$
$h(x) = 4x$
Find $(g - h)(x)$

18) $g(x) = 2x - 3$
$h(x) = x^3 - 2x^2 + 2x$
Find $(g - h)(x)$

19) $f(x) = -3x + 2$
$g(x) = x^2 + 5x$
Find $(f - g)(x)$

20) $g(t) = t - 4$
$h(t) = 2t$
Find $(g \cdot h)(t)$

21) $g(x) = 4x + 5$
$h(x) = x^2 + 5x$
Find $g(x) \cdot h(x)$

22) $g(t) = -2t^2 - 5t$
$h(t) = t + 5$
Find $g(t) \cdot h(t)$

23) $f(x) = x^2 - 5x$
$g(x) = x + 5$
Find $(f + g)(x)$

24) $f(x) = 4x - 4$
$g(x) = 3x^2 - 5$
Find $(f + g)(x)$

25) $g(n) = n^2 + 5$
$f(n) = 3n + 5$
Find $g(n) \div f(n)$

26) $f(x) = 2x + 4$
$g(x) = 4x - 5$
Find $f(x) - g(x)$

27) $g(a) = -2a + 5$
$f(a) = 3a + 5$
Find $(\frac{g}{f})(a)$

28) $g(t) = t^3 + 3t^2$
$h(t) = 3t - 5$
Find $g(t) - h(t)$

29) $h(n) = n^3 + 4n$
$g(n) = 4n + 5$
Find $h(n) + g(n)$

30) $f(x) = 4x + 2$
$g(x) = x^2 + 2x$
Find $f(x) \div g(x)$

31) $g(n) = n^2 - 4n$
$h(n) = n - 5$
Find $g(n^2) \cdot h(n^2)$

32) $g(n) = n + 5$
$h(n) = 2n - 5$
Find $(g \cdot h)(-3n)$

33) $f(x) = 2x$
$g(x) = -3x - 1$
Find $(f + g)(-4 - x)$

34) $g(a) = -2a$
$h(a) = 3a$
Find $g(4n) \div h(4n)$

35) $f(t) = t^2 + 4t$
$g(t) = 4t + 2$
Find $f(t^2) + g(t^2)$

36) $h(n) = 3n - 2$
$g(n) = -3n^2 - 4n$
Find $h(\frac{n}{3}) \div g(\frac{n}{3})$

37) $g(a) = a^3 + 2a$
$h(a) = 3a + 4$
Find $(\frac{g}{h})(-x)$

38) $g(x) = -4x + 2$
$h(x) = x^2 - 5$
Find $g(x^2) + h(x^2)$

39) $f(n) = -3n^2 + 1$
$g(n) = 2n + 1$
Find $(f - g)(\frac{n}{3})$

40) $f(n) = 3n + 4$
$g(n) = n^3 - 5n$
Find $f(\frac{n}{2}) - g(\frac{n}{2})$

41) $f(x) = -4x + 1$
$g(x) = 4x + 3$
Find $(f \circ g)(9)$

42) $g(x) = x - 1$
Find $(g \circ g)(7)$

43) $h(a) = 3a + 3$
$g(a) = a + 1$
Find $(h \circ g)(5)$

44) $g(t) = t + 3$
$h(t) = 2t - 5$
Find $(g \circ h)(3)$

45) $g(x) = x + 4$
$h(x) = x^2 - 1$
Find $(g \circ h)(10)$

46) $f(a) = 2a - 4$
$g(a) = a^2 + 2a$
Find $(f \circ g)(-4)$

47) $f(n) = -4n + 2$
$g(n) = n + 4$
Find $(f \circ g)(9)$

48) $g(x) = 3x + 4$
$h(x) = x^3 + 3x$
Find $(g \circ h)(3)$

49) $g(x) = 2x - 4$
$h(x) = 2x^3 + 4x^2$
Find $(g \circ h)(3)$

50) $g(a) = a^2 + 3$
Find $(g \circ g)(-3)$

51) $g(x) = x^2 - 5x$
$h(x) = 4x + 4$
Find $(g \circ h)(x)$

52) $g(a) = 2a + 4$
$h(a) = -4a + 5$
Find $(g \circ h)(a)$

53) $f(a) = -2a + 2$
$g(a) = 4a$
Find $(f \circ g)(a)$

54) $g(t) = -t - 4$
Find $(g \circ g)(t)$

55) $g(x) = 4x + 4$
$f(x) = x^3 - 1$
Find $(g \circ f)(x)$

56) $f(n) = -2n^2 - 4n$
$g(n) = n + 2$
Find $(f \circ g)(n)$

57) $g(x) = -x + 5$
$f(x) = 2x - 3$
Find $(g \circ f)(x)$

58) $g(t) = t^3 - t$
$f(t) = 3t - 4$
Find $(g \circ f)(t)$

59) $f(t) = 4t + 3$
$g(t) = -4t - 2$
Find $(f \circ g)(t)$

60) $f(x) = 3x - 4$
$g(x) = x^3 + 2x^2$
Find $(f \circ g)(x)$

Answers - Operations on Functions

1) 82

2) 20

3) 46

4) 2

5) 5

6) -30

7) -3

8) 140

9) 1

10) -43

11) 100

12) -74

13) $\frac{1}{5}$

14) 27

15) $-\frac{9}{26}$

16) n^2-2n

17) $-x^3-4x-2$

18) $-x^3+2x^2-3$

19) $-x^2-8x+2$

20) $2t^2-8t$

21) $4x^3+25x^2+25x$

22) $-2t^3-15t^2-25t$

23) x^2-4x+5

24) $3x^2+4x-9$

25) $\frac{n^2+5}{3n+5}$

26) $-2x+9$

27) $\frac{-2a+5}{3a+5}$

28) t^3+3t^2-3t+5

29) n^3+8n+5

30) $\frac{4x+2}{x^2+2x}$

31) $n^6-9n^4+20n^2$

32) $18n^2-15n-25$

33) $x+3$

34) $-\frac{2}{3}$

35) t^4+8t^2+2

36) $\frac{3n-6}{-n^2-4n}$

37) $\frac{-x^3-2x}{-3x+4}$

38) x^4-4x^2-3

39) $\frac{-n^2-2n}{3}$

40) $\frac{32+23n-n^3}{8}$

41) -155

42) 5

43) 21

44) 4

45) 103

46) 12

47) -50

48) 112

49) 176

50) 147

51) $16x^2+12x-4$

52) $-8a+14$

53) $-8a+2$

54) t

55) $4x^3$

56) $-2n^2-12n-16$

57) $-2x+8$

58) $27t^3-108t^2+141t-60$

59) $-16t-5$

60) $3x^3+6x^2-4$