Objective: Combine functions using sum, difference, product, quotient and composition of functions.

Several functions can work together in one larger function. There are 5 common operations that can be performed on functions. The four basic operations on functions are adding, subtracting, multiplying, and dividing. The notation for these functions is as follows.

**Addition**  \((f + g)(x) = f(x) + g(x)\)

**Subtraction**  \((f - g)(x) = f(x) - g(x)\)

**Multiplication**  \((f \cdot g)(x) = f(x)g(x)\)

**Division**  \(\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\)

When we do one of these four basic operations we can simply evaluate the two functions at the value and then do the operation with both solutions

**Example 1.**

\(f(x) = x^2 - x - 2\)
\(g(x) = x + 1\)

Evaluate \(f\) and \(g\) at \(-3\)

\(f(-3) = (-3)^2 - (-3) - 2\)  Evaluate \(f\) at \(-3\)
\(f(-3) = 9 + 3 - 2\)
\(f(-3) = 10\)

\(g(-3) = (-3) + 1\)  Evaluate \(g\) at \(-3\)
\(g(-3) = -2\)

\(f(-3) + g(-3)\)  Add the two functions together
\(10 + (-2)\)  Add
\(8\)  Our Solution

The process is the same regardless of the operation being performed.

**Example 2.**

\(h(x) = 2x - 4\)
\(k(x) = -3x + 1\)  Evaluate \(h\) and \(k\) at 5

Find \((h \cdot k)(5)\)

\(h(5) = 2(5) - 4\)  Evaluate \(h\) at 5
\[ h(5) = 10 - 4 \quad h(5) = 6 \]

\[ k(5) = -3(5) + 1 \quad \text{Evaluate } k \text{ at } 5 \]
\[ k(5) = -15 + 1 \quad k(5) = -14 \]

\[ h(5)k(5) \quad \text{Multiply the two results together} \]
\[ (6)(-14) \quad \text{Multiply} \]
\[ -84 \quad \text{Our Solution} \]

Often as we add, subtract, multiply, or divide functions, we do so in a way that keeps the variable. If there is no number to plug into the equations we will simply use each equation, in parenthesis, and simplify the expression.

**Example 3.**

\[ f(x) = 2x - 4 \]
\[ g(x) = x^2 - x + 5 \]
\[ \text{Write subtraction problem of functions} \]
\[ \text{Find } (f - g)(x) \]

\[ f(x) - g(x) \quad \text{Replace } f(x) \text{ with } (2x - 3) \text{ and } g(x) \text{ with } (x^2 - x + 5) \]
\[ (2x - 4) - (x^2 - x + 5) \quad \text{Distribute the negative} \]
\[ 2x - 4 - x^2 + x - 5 \quad \text{Combine like terms} \]
\[ -x^2 + 3x - 9 \quad \text{Our Solution} \]

The parenthesis are very important when we are replacing \( f(x) \) and \( g(x) \) with a variable. In the previous example we needed the parenthesis to know to distribute the negative.

**Example 4.**

\[ f(x) = x^2 - 4x - 5 \]
\[ g(x) = x - 5 \]
\[ \text{Write division problem of functions} \]
\[ \text{Find } \left( \frac{f}{g} \right)(x) \]

\[ \frac{f(x)}{g(x)} \quad \text{Replace } f(x) \text{ with } (x^2 - 4x - 5) \text{ and } g(x) \text{ with } (x - 5) \]
\[ \frac{(x^2 - 4x - 5)}{(x - 5)} \quad \text{To simplify the fraction we must first factor} \]
\[
\frac{(x - 5)(x + 1)}{(x - 5)} \quad \text{Divide out common factor of } x - 5
\]

\[x + 1 \quad \text{Our Solution}\]

Just as we could substitute an expression into evaluating functions, we can substitute an expression into the operations on functions.

**Example 5.**

\[
f(x) = 2x - 1 \\
g(x) = x + 4 \quad \text{Write as a sum of functions}
\]

Find \((f + g)(x^2)\)

\[
f(x^2) + g(x^2) \quad \text{Replace } x \text{ in } f(x) \text{ and } g(x) \text{ with } x^2
\]

\[
[2(x^2) - 1] + [(x^2) + 4] \quad \text{Distribute the + does not change the problem}
\]

\[
2x^2 - 1 + x^2 + 4 \quad \text{Combine like terms}
\]

\[
3x^2 + 3 \quad \text{Our Solution}
\]

**Example 6.**

\[
f(x) = 2x - 1 \\
g(x) = x + 4 \quad \text{Write as a product of functions}
\]

Find \((f \cdot g)(3x)\)

\[
f(3x)g(3x) \quad \text{Replace } x \text{ in } f(x) \text{ and } g(x) \text{ with } 3x
\]

\[
[2(3x) - 1][3x + 4] \quad \text{Multiply our } 2(3x)
\]

\[
(6x - 1)(3x + 4) \quad \text{FOIL}
\]

\[
18x^2 + 24x - 3x - 4 \quad \text{Combine like terms}
\]

\[
18x^2 + 21x - 4 \quad \text{Our Solution}
\]

The fifth operation of functions is called composition of functions. A composition of functions is a function inside of a function. The notation used for composition of functions is:

\[
(f \circ g)(x) = f(g(x))
\]

To calculate a composition of function we will evaluate the inner function and substitute the answer into the outer function. This is shown in the following example.
Example 7.

\[ a(x) = x^2 - 2x + 1 \]
\[ b(x) = x - 5 \quad \text{Rewrite as a function in function} \]

Find \((a \circ b)(3)\)

\[ a(b(3)) \quad \text{Evaluate the inner function first, } b(3) \]
\[ b(3) = (3) - 5 = -2 \quad \text{This solution is put into } a, a(-2) \]
\[ a(-2) = (-2)^2 - 2(-2) + 1 \quad \text{Evaluate} \]
\[ a(-2) = 4 + 4 + 1 \quad \text{Add} \]
\[ a(-2) = 9 \quad \text{Our Solution} \]

We can also evaluate a composition of functions at a variable. In these problems
we will take the inside function and substitute into the outside function.

Example 8.

\[
\begin{align*}
  f(x) &= x^2 - x \\
  g(x) &= x + 3 \quad \text{Rewrite as a function in function}
\end{align*}
\]

Find \((f \circ g)(x)\)

\[
\begin{align*}
  f(g(x)) &\quad \text{Replace } g(x) \text{ with } x + 3 \\
  f(x + 3) &\quad \text{Replace the variables in } f \text{ with } (x + 3) \\
  (x + 3)^2 - (x + 3) &\quad \text{Evaluate exponent} \\
  (x^2 + 6x + 9) - (x + 3) &\quad \text{Distribute negative} \\
  x^2 + 6x + 9 - x - 3 &\quad \text{Combine like terms} \\
  x^2 + 5x + 6 &\quad \text{Our Solution}
\end{align*}
\]

It is important to note that very rarely is \((f \circ g)(x)\) the same as \((g \circ f)(x)\) as the following example will show, using the same equations, but compositing them in the opposite direction.

Example 9.

\[
\begin{align*}
  f(x) &= x^2 - x \\
  g(x) &= x + 3 \quad \text{Rewrite as a function in function}
\end{align*}
\]

Find \((g \circ f)(x)\)

\[
\begin{align*}
  g(f(x)) &\quad \text{Replace } f(x) \text{ with } x^2 - x \\
  g(x^2 - x) &\quad \text{Replace the variable in } g \text{ with } (x^2 - x) \\
  (x^2 - x) + 3 &\quad \text{Here the parenthesis don’t change the expression} \\
  x^2 - x + 3 &\quad \text{Our Solution}
\end{align*}
\]

World View Note: The term “function” came from Gottfried Wilhelm Leibniz, a German mathematician from the late 17th century.
10.2 Practice - Operations on Functions

Perform the indicated operations.

1) \( g(a) = a^3 + 5a^2 \\ f(a) = 2a + 4 \\ Find \ g(3) + f(3) \)

3) \( g(a) = 3a + 3 \\ f(a) = 2a - 2 \\ Find \ (g + f)(9) \)

5) \( g(x) = x + 3 \\ f(x) = -x + 4 \\ Find \ (g - f)(3) \)

7) \( g(x) = x^2 + 2 \\ f(x) = 2x + 5 \\ Find \ (g - f)(0) \)

9) \( g(t) = t - 3 \\ h(t) = -3t^3 + 6t \\ Find \ g(1) + h(1) \)

11) \( h(t) = t + 5 \\ g(t) = 3t - 5 \\ Find \ (h \cdot g)(5) \)

13) \( h(n) = 2n - 1 \\ g(n) = 3n - 5 \\ Find \ h(0) ÷ g(0) \)

15) \( f(a) = -2a - 4 \\ g(a) = a^2 + 3 \\ Find \ \left(\frac{f}{g}\right)(7) \)

17) \( g(x) = -x^3 - 2 \\ h(x) = 4x \\ Find \ (g - h)(x) \)

19) \( f(x) = -3x + 2 \\ g(x) = x^2 + 5x \\ Find \ (f - g)(x) \)

21) \( g(x) = 4x + 5 \\ h(x) = x^2 + 5x \\ Find \ g(x) \cdot h(x) \)

2) \( f(x) = -3x^2 + 3x \\ g(x) = 2x + 5 \\ Find \ f(-4) ÷ g(-4) \)

4) \( g(x) = 4x + 3 \\ h(x) = x^3 - 2x^2 \\ Find \ (g - h)(-1) \)

6) \( g(x) = -4x + 1 \\ h(x) = -2x - 1 \\ Find \ g(5) + h(5) \)

8) \( g(x) = 3x + 1 \\ f(x) = x^3 + 3x^2 \\ Find \ g(2) \cdot f(2) \)

10) \( f(n) = n - 5 \\ g(n) = 4n + 2 \\ Find \ (f + g)(-8) \)

12) \( g(a) = 3a - 2 \\ h(a) = 4a - 2 \\ Find \ (g + h)(-10) \)

14) \( g(x) = x^2 - 2 \\ h(x) = 2x + 5 \\ Find \ g(-6) + h(-6) \)

16) \( g(n) = n^2 - 3 \\ h(n) = 2n - 3 \\ Find \ (g - h)(n) \)

18) \( g(x) = 2x - 3 \\ h(x) = x^3 - 2x^2 + 2x \\ Find \ (g - h)(x) \)

20) \( g(t) = t - 4 \\ h(t) = 2t \\ Find \ (g \cdot h)(t) \)

22) \( g(t) = -2t^2 - 5t \\ h(t) = t + 5 \\ Find \ g(t) \cdot h(t) \)
23) \( f(x) = x^2 - 5x \)
\( g(x) = x + 5 \)
Find \((f + g)(x)\)

25) \( g(n) = n^2 + 5 \)
\( f(n) = 3n + 5 \)
Find \(g(n) \div f(n)\)

27) \( g(a) = -2a + 5 \)
\( f(a) = 3a + 5 \)
Find \((g^2)(a)\)

29) \( h(n) = n^3 + 4n \)
\( g(n) = 4n + 5 \)
Find \(h(n) + g(n)\)

31) \( g(n) = n^2 - 4n \)
\( h(n) = n - 5 \)
Find \(g(n^2) \cdot h(n^2)\)

33) \( f(x) = 2x \)
\( g(x) = -3x - 1 \)
Find \((f + g)(-4 - x)\)

35) \( f(t) = t^2 + 4t \)
\( g(t) = 4t + 2 \)
Find \(f(t^2) + g(t^2)\)

37) \( g(a) = a^3 + 2a \)
\( h(a) = 3a + 4 \)
Find \((\frac{g}{h})(-x)\)

39) \( f(n) = -3n^2 + 1 \)
\( g(n) = 2n + 1 \)
Find \((f - g)(\frac{n}{3})\)

41) \( f(x) = -4x + 1 \)
\( g(x) = 4x + 3 \)
Find \((f \circ g)(9)\)

43) \( h(a) = 3a + 3 \)
\( g(a) = a + 1 \)
Find \((h \circ g)(5)\)

45) \( g(x) = x + 4 \)
\( h(x) = x^2 - 1 \)
Find \((g \circ h)(10)\)

46) \( f(x) = 4x - 4 \)
\( g(x) = 3x^2 - 5 \)
Find \((f + g)(x)\)

24) \( f(x) = 4x - 4 \)
\( g(x) = 3x^2 - 5 \)
Find \((f + g)(x)\)

26) \( f(x) = 2x + 4 \)
\( g(x) = 4x - 5 \)
Find \(f(x) - g(x)\)

28) \( g(t) = t^3 + 3t^2 \)
\( h(t) = 3t - 5 \)
Find \(g(t) - h(t)\)

30) \( f(x) = 4x + 2 \)
\( g(x) = x^2 + 2x \)
Find \(f(x) \div g(x)\)

32) \( g(n) = n + 5 \)
\( h(n) = 2n - 5 \)
Find \((g \cdot h)(-3n)\)

34) \( g(a) = -2a \)
\( h(a) = 3a \)
Find \(g(4n) \div h(4n)\)

36) \( h(n) = 3n - 2 \)
\( g(n) = -3n^2 - 4n \)
Find \(h(\frac{n}{3}) \div g(\frac{n}{3})\)

38) \( g(x) = -4x + 2 \)
\( h(x) = x^2 - 5 \)
Find \(g(x^2) + h(x^2)\)

40) \( f(n) = 3n + 4 \)
\( g(n) = n^3 - 5n \)
Find \(f(\frac{n}{7}) - g(\frac{n}{7})\)

42) \( g(x) = x - 1 \)
Find \((g \circ g)(7)\)

44) \( g(t) = t + 3 \)
\( h(t) = 2t - 5 \)
Find \((g \circ h)(3)\)

46) \( f(a) = 2a - 4 \)
\( g(a) = a^2 + 2a \)
Find \((f \circ g)(-4)\)
47) \( f(n) = -4n + 2 \)
\hspace{1cm} g(n) = n + 4
\hspace{1cm} \text{Find} \ (f \circ g)(9)

48) \( g(x) = 3x + 4 \)
\hspace{1cm} h(x) = x^3 + 3x
\hspace{1cm} \text{Find} \ (g \circ h)(3)

49) \( g(x) = 2x - 4 \)
\hspace{1cm} h(x) = 2x^3 + 4x^2
\hspace{1cm} \text{Find} \ (g \circ h)(3)

50) \( g(a) = a^2 + 3 \)
\hspace{1cm} \text{Find} \ (g \circ g)(-3)

51) \( g(x) = x^2 - 5x \)
\hspace{1cm} h(x) = 4x + 4
\hspace{1cm} \text{Find} \ (g \circ h)(x)

52) \( g(a) = 2a + 4 \)
\hspace{1cm} h(a) = -4a + 5
\hspace{1cm} \text{Find} \ (g \circ h)(a)

53) \( f(a) = -2a + 2 \)
\hspace{1cm} g(a) = 4a
\hspace{1cm} \text{Find} \ (f \circ g)(a)

54) \( g(t) = -t - 4 \)
\hspace{1cm} \text{Find} \ (g \circ g)(t)

55) \( g(x) = 4x + 4 \)
\hspace{1cm} f(x) = x^3 - 1
\hspace{1cm} \text{Find} \ (g \circ f)(x)

56) \( f(n) = -2n^2 - 4n \)
\hspace{1cm} g(n) = n + 2
\hspace{1cm} \text{Find} \ (f \circ g)(n)

57) \( g(x) = -x + 5 \)
\hspace{1cm} f(x) = 2x - 3
\hspace{1cm} \text{Find} \ (g \circ f)(x)

58) \( g(t) = t^3 - t \)
\hspace{1cm} f(t) = 3t - 4
\hspace{1cm} \text{Find} \ (g \circ f)(t)

59) \( f(t) = 4t + 3 \)
\hspace{1cm} g(t) = -4t - 2
\hspace{1cm} \text{Find} \ (f \circ g)(t)

60) \( f(x) = 3x - 4 \)
\hspace{1cm} g(x) = x^3 + 2x^2
\hspace{1cm} \text{Find} \ (f \circ g)(x)

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### Answers - Operations on Functions

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57) $-2x + 8$
58) $27t^3 - 108t^2 + 141t - 60$
59) $-16t - 5$
60) $3x^3 + 6x^2 - 4$

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