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Solving Linear Equations - Absolute Value

Objective: Solve linear absolute value equations.

When solving equations with absolute value we can end up with more than one possible answer. This is because what is in the absolute value can be either negative or positive and we must account for both possibilities when solving equations. This is illustrated in the following example.

Example 1.

$$|x| = 7 \quad \text{Absolute value can be positive or negative}$$
$$x = 7 \text{ or } x = -7 \quad \text{Our Solution}$$

Notice that we have considered two possibilities, both the positive and negative. Either way, the absolute value of our number will be positive 7.

World View Note: The first set of rules for working with negatives came from 7th century India. However, in 1758, almost a thousand years later, British mathematician Francis Maseres claimed that negatives "Darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple."

When we have absolute values in our problem it is important to first isolate the absolute value, then remove the absolute value by considering both the positive and negative solutions. Notice in the next two examples, all the numbers outside of the absolute value are moved to the other side first before we remove the absolute value bars and consider both positive and negative solutions.

Example 2.

$$5 + |x| = 8 \quad \text{Notice absolute value is not alone}$$
$$\underline{-5 \qquad -5} \quad \text{Subtract 5 from both sides}$$
$$|x| = 3 \quad \text{Absolute value can be positive or negative}$$
$$x = 3 \text{ or } x = -3 \quad \text{Our Solution}$$

Example 3.

$$-4|x| = -20 \quad \text{Notice absolute value is not alone}$$
$$\overline{-4} \qquad \overline{-4} \quad \text{Divide both sides by } -4$$

$$|x| = 5 \quad \text{Absolute value can be positive or negative}$$
$$x = 5 \text{ or } x = -5 \quad \text{Our Solution}$$

Notice we never combine what is inside the absolute value with what is outside the absolute value. This is very important as it will often change the final result to an incorrect solution. The next example requires two steps to isolate the absolute value. The idea is the same as a two-step equation, add or subtract, then multiply or divide.

Example 4.

$$
\begin{array}{rl}
5|x| - 4 = 26 & \text{Notice the absolute value is not alone} \\
\underline{\mathbf{+4 \; +4}} & \text{Add 4 to both sides} \\
5|x| = 30 & \text{Absolute value still not alone} \\
\overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \text{Divide both sides by 5} \\
|x| = 6 & \text{Absolute value can be positive or negative} \\
x = 6 \text{ or } x = -6 & \text{Our Solution}
\end{array}
$$

Again we see the same process, get the absolute value alone first, then consider the positive and negative solutions. Often the absolute value will have more than just a variable in it. In this case we will have to solve the resulting equations when we consider the positive and negative possibilities. This is shown in the next example.

Example 5.

$$
\begin{array}{rl}
|2x - 1| = 7 & \text{Absolute value can be positive or negative} \\
2x - 1 = 7 \text{ or } 2x - 1 = -7 & \text{Two equations to solve}
\end{array}
$$

Now notice we have two equations to solve, each equation will give us a different solution. Both equations solve like any other two-step equation.

$$
\begin{array}{rcr}
2x - 1 = 7 & & 2x - 1 = -7 \\
\underline{\mathbf{+1 \; +1}} & & \underline{\mathbf{+1 \; +1}} \\
2x = 8 & \text{or} & 2x = -6 \\
\overline{\mathbf{2}} \quad \overline{\mathbf{2}} & & \overline{2} \quad \overline{2} \\
x = 4 & & x = -3
\end{array}
$$

Thus, from our previous example we have two solutions, $x = 4$ or $x = -3$.

Again, it is important to remember that the absolute value must be alone first before we consider the positive and negative possibilities. This is illustrated in below.

Example 6.

$$2 - 4|2x + 3| = -18$$

To get the absolute value alone we first need to get rid of the 2 by subtracting, then divide by -4. Notice we cannot combine the 2 and -4 becuase they are not like terms, the -4 has the absolute value connected to it. Also notice we do not distribute the -4 into the absolute value. This is because the numbers outside cannot be combined with the numbers inside the absolute value. Thus we get the absolute value alone in the following way:

$$\begin{array}{rl} 2 - 4|2x + 3| = -18 & \text{Notice absolute value is not alone} \\ \underline{\mathbf{-2} \qquad\qquad\qquad \mathbf{-2}} & \text{Subtract 2 from both sides} \\ -4|2x + 3| = -20 & \text{Absolute value still not alone} \\ \overline{\mathbf{-4}} \qquad\quad \overline{\mathbf{-4}} & \text{Divide both sides by } -4 \\ |2x + 3| = 5 & \text{Absoloute value can be positive or negative} \\ 2x + 3 = 5 \text{ or } 2x + 3 = -5 & \text{Two equations to solve} \end{array}$$

Now we just solve these two remaining equations to find our solutions.

$$\begin{array}{rcr} 2x + 3 = 5 & & 2x + 3 = -5 \\ \underline{\mathbf{-3}\ \mathbf{-3}} & & \underline{\mathbf{-3}\ \ \mathbf{-3}} \\ 2x = 2 & \text{or} & 2x = -8 \\ \overline{\mathbf{2}}\ \ \overline{\mathbf{2}} & & \overline{\mathbf{2}}\ \ \ \overline{\mathbf{2}} \\ x = 1 & & x = -4 \end{array}$$

We now have our two solutions, $x = 1$ and $x = -4$.

As we are solving absolute value equations it is important to be aware of special cases. Remember the result of an absolute value must always be positive. Notice what happens in the next example.

Example 7.

$$\begin{array}{rl} 7+|2x-5|=4 & \text{Notice absolute value is not alone} \\ \mathbf{-7} \qquad\quad \mathbf{-7} & \text{Subtract 7 from both sides} \\ \hline |2x-5|=-3 & \text{Result of absolute value is negative!} \end{array}$$

Notice the absolute value equals a negative number! This is impossible with absolute value. When this occurs we say there is **no solution** or $\varnothing$.

One other type of absolute value problem is when two absolute values are equal to eachother. We still will consider both the positive and negative result, the difference here will be that we will have to distribute a negative into the second absolute value for the negative possibility.

Example 8.

$$\begin{array}{cl} |2x-7|=|4x+6| & \text{Absolute value can be positive or negative} \\ 2x-7=4x+6 \quad \text{or} \quad 2x-7=-(4x+6) & \text{make second part of second equation negative} \end{array}$$

Notice the first equation is the positive possibility and has no significant difference other than the missing absolute value bars. The second equation considers the negative possibility. For this reason we have a negative in front of the expression which will be distributed through the equation on the first step of solving. So we solve both these equations as follows:

$$\begin{array}{ccc} \begin{array}{r} 2x-7=4x+6 \\ \underline{\mathbf{-2x} \quad \mathbf{-2x}} \\ -7=2x+6 \\ \underline{\mathbf{-6} \qquad \mathbf{-6}} \\ -13=2x \\ \overline{\mathbf{2}} \quad \overline{\mathbf{2}} \\ \frac{-13}{2}=x \end{array} & \text{or} & \begin{array}{r} 2x-7=-(4x+6) \\ 2x-7=-4x-6 \\ \underline{\mathbf{+4x} \quad \mathbf{+4x}} \\ 6x-7=-6 \\ \underline{\mathbf{+7} \quad \mathbf{+7}} \\ 6x=1 \\ \overline{\mathbf{6}} \quad \overline{\mathbf{6}} \\ x=\frac{1}{6} \end{array} \end{array}$$

This gives us our two solutions, $x=\frac{-13}{2}$ or $x=\frac{1}{6}$.

1.6 Practice - Absolute Value Equations

Solve each equation.

1) $|x| = 8$

2) $|n| = 7$

3) $|b| = 1$

4) $|x| = 2$

5) $|5 + 8a| = 53$

6) $|9n + 8| = 46$

7) $|3k + 8| = 2$

8) $|3 - x| = 6$

9) $|9 + 7x| = 30$

10) $|5n + 7| = 23$

11) $|8 + 6m| = 50$

12) $|9p + 6| = 3$

13) $|6 - 2x| = 24$

14) $|3n - 2| = 7$

15) $-7|-3 - 3r| = -21$

16) $|2 + 2b| + 1 = 3$

17) $7|-7x - 3| = 21$

18) $\frac{|-4 - 3n|}{4} = 2$

19) $\frac{|-4b - 10|}{8} = 3$

20) $8|5p + 8| - 5 = 11$

21) $8|x + 7| - 3 = 5$

22) $3 - |6n + 7| = -40$

23) $5|3 + 7m| + 1 = 51$

24) $4|r + 7| + 3 = 59$

25) $3 + 5|8 - 2x| = 63$

26) $5 + 8|-10n - 2| = 101$

27) $|6b - 2| + 10 = 44$

28) $7|10v - 2| - 9 = 5$

29) $-7 + 8|-7x - 3| = 73$

30) $8|3 - 3n| - 5 = 91$

31) $|5x + 3| = |2x - 1|$

32) $|2 + 3x| = |4 - 2x|$

33) $|3x - 4| = |2x + 3|$

34) $|\frac{2x - 5}{3}| = |\frac{3x + 4}{2}|$

35) $|\frac{4x - 2}{5}| = |\frac{6x + 3}{2}|$

36) $|\frac{3x + 2}{2}| = |\frac{2x - 3}{3}|$

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Answers to Absolute Value Equations

1) $8, \ -8$

2) $7, -7$

3) $1, -1$

4) $2, \ -2$

5) $6, \ -\frac{29}{4}$

6) $\frac{38}{9}, -6$

7) $-2, -\frac{10}{3}$

8) $-3, 9$

9) $3, \ -\frac{39}{7}$

10) $\frac{16}{5}, -6$

11) $7, \ -\frac{29}{3}$

12) $-\frac{1}{3}, -1$

13) $-9, 15$

14) $3, -\frac{5}{3}$

15) $-2, 0$

16) $0, \ -2$

17) $-\frac{6}{7}, 0$

18) $-4, \frac{4}{3}$

19) $-\frac{17}{2}, \frac{7}{2}$

20) $-\frac{6}{5}, -2$

21) $-6, -8$

22) $6, -\frac{25}{3}$

23) $1, \ -\frac{13}{7}$

24) $7, -21$

25) $-2, 10$

26) $-\frac{7}{5}, 1$

27) $6, \ -\frac{16}{3}$

28) $\frac{2}{5}, 0$

29) $-\frac{13}{7}, 1$

30) $-3, 5$

31) $-\frac{4}{3}, -\frac{2}{7}$

32) $-6, \frac{2}{5}$

33) $7, \frac{1}{5}$

34) $-\frac{22}{5}, -\frac{2}{13}$

35) $-\frac{19}{22}, -\frac{11}{38}$

36) $0, -\frac{12}{5}$